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CONFORMAL LIE ALGEBRAS VIA DEFORMATION THEORY

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Abstract. We discuss possible notions of conformal Lie algebras, paying particular attention to graded conformal Lie algebras with d-dimensional space isotropy: namely, those with a $\mathfrak{co}(d)$ subalgebra acting in a prescribed way on the additional generators. We classify those Lie algebras up to isomorphism for all $d \geqslant 2$ following the same methodology used recently to classify kinematical Lie algebras, as deformations of the "most abelian" graded conformal algebra. We find 17 isomorphism classes of Lie algebras for $d \neq 3$ and 23 classes for $d = 3$. Lie algebra contractions define a partial order in the set of isomorphism classes and this is illustrated via the corresponding Hesse diagram. The only metric graded conformal Lie algebras are the simple Lie algebras, isomorphic to either $\mathfrak{so}(d+1, 2)$ or $\mathfrak{so}(d+2, 1)$. We also work out the central extensions of the graded conformal algebras and study their invariant inner products. We find that central extensions of a Lie algebra in $d = 3$ and two Lie algebras in $d = 2$ are metric. We then discuss several other notions of conformal Lie algebras (generalised conformal, Schrödinger and Lifshitz Lie algebras) and we present some partial results on their classification. We end with some open problems suggested by our results.

Contents

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1. Introduction

In a recent series of papers [1, 2, 3] we classified kinematical Lie algebras in arbitrary dimensions via deformation theory. This extends to arbitrary dimension the classic results for $3+1$ dimensions by Bacry and Nuyts [4], following up form the earlier work of Bacry and Lévy-Leblond [5]. A natural extension of the classification problem of kinematical Lie algebras is the classification of conformal Lie algebras, but one first has to decide what a conformal Lie algebra is.

Just like not all kinematical Lie algebras act by isometries on a pseudo-riemannian spacetime, we do not expect all conformal Lie algebras (regardless of the definition) to act by conformal transformations on a conformal manifold. But by the same token, just like kinematical Lie algebras generalise the Lie algebras of isometries of the maximally symmetric riemannian and lorentzian spaces, whichever definition of conformal Lie algebra one adopts, one would expect that the Lie algebra of conformal Killing vector fields on one of these spaces qualifies as a conformal Lie algebra; although as we shall see there are notions closely related to conformal Lie algebras (e.g., Lifshitz and Schrödinger algebras) which deviate from this requirement.

The Lie algebra of conformal Killing vector fields of (d+1)-dimensional euclidean space and Minkowski spacetime (or indeed of any other simply-connected conformally flat riemannian or lorentzian manifold of the same dimension) is isomorphic to $\mathfrak{so}(d+2,1)$ in the riemannian case or $\mathfrak{so}(d+1,2)$ in the lorentzian case. We let $J_{\mu\nu}$ denote the generators, where the index μ decomposes into $(a,+,-,0)$, where $a=1,\dots,d$. The inner product $\eta_{\mu\nu}$ has components

$$\eta_{ab}=\delta_{ab}, \qquad \eta_{+-}=1 \qquad \text{and} \qquad \eta_{00}=\varepsilon, \tag{1}$$

with $\varepsilon=\pm1$ in the riemannian and lorentzian cases, respectively. Then the generators $J_{\mu\nu}$, which satisfy the Lie brackets

$$[J_{\mu\nu},J_{\rho\sigma}]=\eta_{\nu\rho}J_{\mu\sigma}-\eta_{\mu\rho}J_{\nu\sigma}-\eta_{\nu\sigma}J_{\mu\rho}+\eta_{\mu\sigma}J_{\nu\rho}, \tag{2}$$

break up into J_{ab}, $D:=J_{+-}$, $V_{0\,a}:=J_{a0}$, $V_{\pm\,a}:=J_{a\pm}$ and $S_\pm:=J_{0\pm}$, with Lie brackets

$$\begin{aligned}
[J_{ab},J_{cd}]&=\delta_{bc}J_{ad}-\delta_{ac}J_{bd}-\delta_{bd}J_{ac}+\delta_{ad}J_{bc} & [D,S_\pm]&=\pm S_\pm\\
[J_{ab},V_{\pm\,c}]&=\delta_{bc}V_{\pm\,a}-\delta_{ac}V_{\pm\,b} & [J_{ab},S_\pm]&=0\\
[J_{ab},V_{0\,c}]&=\delta_{bc}V_{0\,a}-\delta_{ac}V_{0\,b} & [J_{ab},D]&=0\\
[D,V_{\pm\,a}]&=\pm V_{\pm\,a} & [D,V_{0\,a}]&=0.
\end{aligned} \tag{3}$$

and, in addition,

$$\begin{aligned}
[S_\pm,V_{0\,a}]&=V_{\pm\,a} & & & [V_{0\,a},V_{0\,b}]&=-\varepsilon J_{ab}\\
[S_\pm,V_{\mp\,a}]&=V_{0\,a} & &\text{and} & [V_{0\,a},V_{\pm\,b}]&=\varepsilon\delta_{ab}S_\pm\\
[S_+,S_-]&=D & & & [V_{+\,a},V_{-\,b}]&=\varepsilon(J_{ab}+\delta_{ab}D).
\end{aligned} \tag{4}$$

What properties of the above Lie algebras shall we take as characterising the notion of a "conformal Lie algebra"?

The obvious geometrical answer is that the above Lie algebras are isomorphic to Lie algebras of conformal vector fields on pseudo-riemannian manifolds. The Lie algebra of conformal Killing vectors in a pseudo-riemannian manifold has a precise algebraic structure shared with the Lie algebra of isometries and, more generally, the Killing superalgebra of a supergravity background [6]. Conformal Killing vectors share, with Killing vectors and Killing spinors in some supergravity theories, the property that they are parallel sections of a vector bundle with connection. The connection in the conformal case is the one that gives rise to the notion of conformal Killing transport [7], an extension to conformal Killing vectors of earlier results of Kostant's [8] for Killing vectors. All these (super)algebras have in common that they are filtered deformations of graded subalgebras of a graded Lie (super)algebra: the (super)algebra associated to the "flat" model for the geometry. For conformal Killing transport, the flat model is that of any (simply-connected) conformally flat manifold which, in the present context, would be one of the simple conformal Lie algebras above. Of course, if we insist on the dimension being $\frac{1}{2}(d+3)(d+2)$, then the algebra *is* that of the flat model, but we could drop this requirement and insist only that it should contain an $\mathfrak{so}(d)$ subalgebra and perhaps that it acted transitively. This is an interesting problem which we will not address in this paper.

Although this property of acting like conformal transformations on a pseudo-riemannian manifold is the property dictated by geometric orthodoxy, it is a sense too narrow. Just like not all kinematical Lie algebras are isomorphic to a Lie algebra of isometries in a pseudo-riemannian manifold, we do not wish to restrict attention to Lie algebras of conformal Killing vectors. Kinematical Lie algebras with d-dimensional space isotropy, for $d\geqslant 2$, are symmetries of non-metrical structures such as galilean (or Newton–Cartan), carrollian and aristotelian [9]. By analogy with the case of lorentzian and riemannian spacetimes, one could define a *conformal vector field* to be one which generates conformal rescalings of the relevant structures. Since Newton–Cartan and carrollian structures are degenerate, these conformal vector fields typically span an infinite-dimensional Lie algebra, which in at least one carrollian case has been shown to coincide with the BMS algebra [10].

Another property of the above simple conformal Lie algebras, and one that forms the kinematical basis for the AdS/CFT correspondence, is that they are isomorphic to the Lie algebra of isometries of a lorentzian manifold in one higher dimension. This suggests re-interpreting kinematical Lie algebras with $(d+1)$-dimensional space isotropy as *holographic conformal algebras* with d-dimensional space isotropy.

Moving away from the geometric conformal algebras, we may instead concentrate on algebraic properties. At the surface lies the observation that the simple conformal algebras above have a Lie subalgebra isomorphic to $\mathfrak{so}(d)$, relative to which the additional generators transform as 3 copies of the vector d-dimensional representation and 3 copies of the scalar one-dimensional representation. We shall call such Lie algebras *generalised conformal algebras* (see Definition 2); although it is not clear that they have the right to be called conformal. We consider them because they form a large class of Lie algebras which encompass many of the conformal Lie algebras we consider in this paper.

More closely related to conformality is the fact that the simple conformal algebras above have a Lie subalgebra isomorphic to $\mathfrak{co}(d) = \mathfrak{so}(d) \oplus \mathbb{R}D$, where the adjoint action of the *dilatation* D defines a $\mathbb{Z}$-grading, where J_{ab}, V_0 and D have degree 0, and $V_\pm$ and $S_\pm$ in degree ±1. In this paper we will take this gradation by "conformal weight" to be the defining property of a conformal algebra. We will not just demand that the Lie algebra be graded by D, but we are also specifying the conformal weights.

We may generalise this class of Lie algebras along at least two directions. Firstly, we can still demand on the $\mathfrak{co}(d)$ subalgebra with D a grading element, but allowing for arbitrary conformal weights. This leads to the notion a *generalised Lifshitz algebra* (see Definition 3), discussed briefly in Section 7, where as a first step in their classification, we classify possible $\mathbb{Z}$-gradings on kinematical Lie algebras.

The second generalisation is to demand that the dilatation D should be part of an $\mathfrak{sl}(2,\mathbb{R})$ subalgebra. In other words, the $\mathfrak{co}(d)$-subalgebra extends to an $\mathfrak{so}(d)\oplus\mathfrak{sl}(2,\mathbb{R})$ subalgebra in such a way that the additional generators transform according to a representation $\mathbb{V}\otimes E$ of $\mathfrak{so}(d)\oplus\mathfrak{sl}(2,\mathbb{R})$, where $\mathbb{V}$ is the d-dimensional vector representation of $\mathfrak{so}(d)$ and E is a representation of $\mathfrak{sl}(2,\mathbb{R})$ of dimension 2 or 3. In the case where E is the fundamental representation (so $\dim E = 2$) we will show that for $d \geqslant 3$ the Lie algebra is unique up to isomorphism and admits a unique central extension which is isomorphic to the Schrödinger Lie algebra. In general we shall call the central extensions of such Lie algebras *generalised Schrödinger algebras* (see Definition 4) and we will discuss their classification in Section 8.

We shall now define the main characters in this paper: the graded conformal Lie algebras.

Let $d \geqslant 2$. Recall that $\mathfrak{so}(d)$ is the Lie algebra of skew-symmetric linear transformations of d-dimensional euclidean space and $\mathfrak{co}(d) = \mathfrak{so}(d) \oplus \mathbb{R}D$ is the extension of $\mathfrak{so}(d)$ by dilatations.

Definition 1. By a **graded conformal algebra** (with d-dimensional space isotropy) we mean a real Lie algebra $\mathfrak{g}$ satisfying the following properties

1. $\mathfrak{g}$ has a subalgebra $\mathfrak{h} \cong \mathfrak{co}(d)$, and
2. under the adjoint action of $\mathfrak{h}$, $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{V}_+ \oplus \mathbb{V}_- \oplus \mathbb{V}_0 \oplus \mathbb{S}_+ \oplus \mathbb{S}_-$, where $\mathbb{V}_\pm, \mathbb{V}_0$ are vectors under $\mathfrak{so}(d)$ and have weights ±1 and 0, respectively, under D, whereas $\mathbb{S}_\pm$ are scalars under $\mathfrak{so}(d)$ and have weights ±1 under D.

In other words, a graded conformal algebra is the real span of $J_{ab} = -J_{ba}$, $V_{\pm a}$, V_{0a}, $S_\pm$, with $a, b \in \{1, \dots, d\}$, satisfying equation (3) and any other Lie brackets not mentioned here, subject only to the axioms of a Lie algebra.

Notice that the simple conformal Lie algebras whose brackets are given by (3) and (4) is not just $\mathbb{Z}$-graded, but actually $(\mathbb{Z}_2 \times \mathbb{Z})$-graded, with $J_{ab}, D, S_\pm$ even under $\mathbb{Z}_2$ and $V_{\pm a}, V_{0a}$ odd under $\mathbb{Z}_2$. This parity is maintained in all graded conformal algebras in $d \neq 3$, but not necessarily in $d = 3$ due to the existence of the parity-violating vector product in $\mathbb{R}^3$.

All graded conformal Lie algebras share the brackets (3) and are distinguished by the additional brackets between the generators $V_{\pm a}$, V_{0a}, $S_\pm$. If these additional brackets are zero, we have the **static graded conformal algebra**, which we will denote $\mathfrak{g}$ from now on. Any other graded conformal algebra is, by definition, a deformation of $\mathfrak{g}$ and this is the approach we will take towards the classification. As in the case of kinematical Lie algebras, the problem depends on the value of d, the case $d \geqslant 4$ being generic, whereas $d \leqslant 3$ needs special care. In this paper we will deal with $d \geqslant 2$; although the discussion breaks up into several cases: $d \geqslant 5$, $d = 4$, $d = 3$ and $d = 2$. We refer to [1] for the methodology and to [3] for the rationale of working with complex Lie algebras in $d = 2$.

This paper is organised as follows. In Section 2 we discuss the deformations of the static graded conformal algebra for $d \geqslant 4$, which is the generic range. The bulk of the discussion is about $d \geqslant 5$, but in Section 2.5 we simply observe that the generic case also includes $d = 4$ since there is no substantial change in the calculations. All that happens in $d = 4$ is that $\mathfrak{so}(4)$ is not simple, but this does not change the results. After introducing the deformation complex in Section 2.1, we calculate the second cohomology and thus determine the infinitesimal deformations in Section 2.2. In Section 2.3 we work out the obstructions to integrating the infinitesimal deformations and after solving the resulting system of quadratic equations, in Section 2.4 we exploit the action of automorphisms of $\mathfrak{g}$ in order to classify the isomorphism classes of deformations. The results are summarised in Section 2.6 and particularly in Table 6. In Section 2.7 we show that the only metric Lie algebras in this classification are the simple conformal algebras: $\mathfrak{so}(d+1,2)$ and $\mathfrak{so}(d+2,1)$. In Section 2.8 we show how all graded conformal algebras for $d \geqslant 4$ can be obtained as contractions of the simple conformal algebras and we exhibit explicit contractions for each case.

In Section 3 we discuss the case of $d = 3$, which is substantially different due to the existence of an $\mathfrak{so}(3)$-invariant vector product (or, equivalently, the Levi-Civita symbol ϵ_{abc}) which can now appear in brackets. The panorama of graded conformal algebras in $d = 3$ is somewhat richer than in $d \geqslant 4$, resulting in new graded conformal algebras which have no analogue in $d \geqslant 4$. In Section 3.1 we introduce the differential complex, which is larger than the one for $d \geqslant 4$. Its second cohomology is computed in Section 3.2 and in this way determine the space of infinitesimal deformations. The obstructions to integrability are determined in Section 3.3. There we see that the solution of the integrability conditions leads to several branches of solutions: one of them being the $d = 3$ analogue of the $d \geqslant 4$ deformations. The other branches are unique to $d = 3$ and their study concludes in Section 3.4 with the determination of the isomorphism classes of Lie algebras for $d = 3$. The graded conformal algebras unique to $d = 3$ are listed in Table 10. In Section 3.5 we discuss the contractions and we summarise the results in Figure 1, which illustrates the Hasse diagram of the partial order defined by contraction on the set of isomorphism classes of graded conformal algebras.

In Section 4 we discuss the case of $d = 2$, which although superficially different from $d > 2$ actually results in formally the same classification as $d \geqslant 4$. The complex for $d = 2$ is again different from the generic d due to the existence of the $\mathfrak{so}(2)$-invariant symplectic structure on the vector representation. This manifests itself in a larger endomorphism ring (namely, $\mathbb{C}$ as opposed to $\mathbb{R}$) and it is therefore convenient to work with a complexified Lie algebra, while ensuring that at every moment the deformations are real. In Section 4.1 we introduce the complex Lie algebra whose deformation complex is described in Section 4.2. Infinitesimal deformations are determined in Section 4.3 and the obstructions in Section 4.4. Finally, the isomorphism classes of Lie algebras are determined in Section 4.5, with results which are no different from those of $d \geqslant 4$. This is in sharp contrast with the case of kinematical Lie algebras [3], where the panorama in $d = 2$ was substantially richer than for generic d.

In Section 5 we work out the central extensions of the graded conformal algebras. Again, the results depend on d. In Section 5.1 we classify the universal central extensions for the Lie algebras in Tables 6 and 10. The results are contained in Table 17. In Section 5.2 we investigate whether any central extension admits an invariant inner product. We find that one Lie algebra (GCA_{16}), which exists only for $d = 3$, has a metric central extension. In Section 5.3 we classify the universal central extensions for the $d = 2$ avatars of the Lie algebras in Table 6 and summarise the results in Table 18. The metricity of these central extensions is investigated in Section 5.4, where it is found that two $d = 2$ Lie algebras admit metric central extensions: GCA_1 and GCA_8.

In Section 6 we discuss a generalisation of the notion of graded conformal algebras, where we dispense with the grading. It is questionable to consider them as conformal algebras at all, but they are introduce since they are a large class of algebras with the right spectrum of generators, which encompass many of the other kinds of algebras discussed in this paper. We will not solve the classification problem here, but for $d \geqslant 4$ we will write down the most general deformation and the conditions for integrability, but will not solve them in general.

In Section 7 we discuss generalisations of the Lifshitz algebra (extended by boosts). To a first approximation, these Lie algebras are essentially a graded kinematical Lie algebra extended by the grading element. We classify the consistent gradings of kinematical Lie algebras where $\mathfrak{so}(d)$ is in degree zero, a result summarised in Table 21.

In Section 8 we discuss generalisations of the Schrödinger algebra, which can be understood as a central extension of a conformal algebra (which might be missing a vectorial generator). We define generalised Schrödinger algebras as those containing an $\mathfrak{so}(d) \oplus \mathfrak{sl}(2,\mathbb{R})$ subalgebra, relative to which the vectorial generators transform according to $V \otimes E$, where V is the vector representation of $\mathfrak{so}(d)$ and E is a representation of $\mathfrak{sl}(2,\mathbb{R})$ of dimension 2 or 3. We classify them for $d \geqslant 3$ when $\dim E = 2$ and for $d \geqslant 4$ when $\dim E = 3$.

Finally, Section 9 we summarise the results and make some general comments.

2. Deformations for $d \geqslant 4$

Let $d \geqslant 5$ to begin with. We will let $\mathbb{W}$ denote the real vector space spanned by $V_{\pm a}, V_{0a}, S_\pm$ and let $\mathbb{W}^*$ denote its dual, with canonical dual basis $\upsilon^\pm_a, \upsilon^0_a, \sigma^\pm$. (Because of the existence of the $\mathfrak{so}(d)$-invariant δ_{ab} we are free to identify vector subscripts and superscripts, explaining why we write υ^0_a and not, say, $\upsilon^{0\,a}$.) Recall that $\mathfrak{h} \cong \mathfrak{co}(d)$ is the Lie subalgebra spanned by J_{ab} and D.

2.1. The deformation complex.

The differential complex for deformations of the static graded conformal algebra is denoted $(C^\bullet, \partial)$. The cochains are those of the static conformal algebra $\mathfrak{g}$ relative to $\mathfrak{h}$ and with values in the adjoint module $\mathfrak{g}$:

$$C^p := C^p(\mathfrak{g},\mathfrak{h};\mathfrak{g}) \cong \operatorname{Hom}(\Lambda^p\mathbb{W},\mathfrak{g})^{\mathfrak{h}} \cong (\Lambda^p\mathbb{W}^* \otimes \mathfrak{g})^{\mathfrak{h}}, \tag{5}$$

which is the space of $\mathfrak{h}$-equivariant skewsymmetric p-multilinear maps $\mathbb{W} \times \cdots \times \mathbb{W} \to \mathfrak{g}$, and the differential $\partial : C^p \to C^{p+1}$ is defined on generators by

$$\begin{aligned} \partial J_{ab} &= \upsilon_a^+ V_{+\,b} - \upsilon_b^+ V_{+\,a} + \upsilon_a^- V_{-\,b} - \upsilon_b^- V_{-\,a} + \upsilon_a^0 V_{0\,b} - \upsilon_b^0 V_{0\,a} \\ \partial D &= -\upsilon_a^+ V_{+\,a} + \upsilon_a^- V_{-\,b} - \sigma^+ S_+ + \sigma^- S_-, \end{aligned} \tag{6}$$

and zero elsewhere, where we have omitted $\otimes$ in expressions such as $\sigma^+ \otimes S_+$, et cetera. The cochains do not have any legs on $\mathfrak{h}$ because we do not wish to deform any of the brackets involving J_{ab} or D. The $\mathfrak{h}$-equivariance ensures that the Jacobi identity involving one generator in $\mathfrak{h}$ is satisfied.

Deformation theory only requires only C^p for $p = 1, 2, 3$, of which the first two are

$$\begin{aligned} C^1 &= \operatorname{span}_{\mathbb{R}}(\upsilon^+ V_+, \upsilon^- V_-, \upsilon^0 V_0, \sigma^+ S_+, \sigma^- S_-) \\ C^2 &= \operatorname{span}_{\mathbb{R}}(\upsilon^+\upsilon^- J, \upsilon^+\upsilon^- D, \tfrac{1}{2}\upsilon^0\upsilon^0 J, \sigma^+\sigma^- D, \upsilon^0\upsilon^+ S_+, \upsilon^0\upsilon^- S_-, \sigma^+\upsilon^- V_0, \sigma^-\upsilon^+ V_0, \sigma^+\upsilon^0 V_+, \sigma^-\upsilon^0 V_-), \end{aligned} \tag{7}$$

where we have introduced shorthand notations such as

$$\upsilon^+ V_+ := \upsilon_a^+ \otimes V_{+\,a} \qquad \text{and} \qquad \upsilon^+\upsilon^- J := \upsilon_a^+ \wedge \upsilon_b^- \otimes J_{ab}. \tag{8}$$

2.2. **Infinitesimal deformations.** Infinitesimal deformations are classified by H^2, which, since $\partial : C^1 \to C^2$ is the zero map, is given by $H^2 = \ker\left(\partial : C^2 \to C^3\right)$.

The differential $\partial : C^2 \to C^3$ has nonzero components

$$\begin{aligned} \partial(\sigma^+\sigma^- D) &= \sigma^+\sigma^-\upsilon^- V_- - \sigma^+\sigma^-\upsilon^+ V_+ \\ \partial(\upsilon^+\upsilon^- D) &= -\upsilon^+\upsilon^-\upsilon^+ V_+ + \upsilon^+\upsilon^-\upsilon^- V_- - \sigma^+\upsilon^+\upsilon^- S_+ + \sigma^-\upsilon^+\upsilon^- S_- \\ \partial(\upsilon^+\upsilon^- J) &= \upsilon^+\upsilon^-\upsilon^+ V_+ - \upsilon^+\upsilon^-\upsilon^- V_- + \upsilon^0\upsilon^+\upsilon^- V_0 + \upsilon^0\upsilon^-\upsilon^+ V_0 \\ \partial(\tfrac{1}{2}\upsilon^0\upsilon^0 J) &= -\upsilon^0\upsilon^+\upsilon^0 V_+ - \upsilon^0\upsilon^-\upsilon^0 V_-, \end{aligned} \tag{9}$$

where the abbreviated notation is such that $\upsilon^+\upsilon^-\upsilon^+ V_+ := \upsilon_a^+ \wedge \upsilon_a^- \wedge \upsilon_b^+ \otimes V_{+\,b}$, et cetera. This implies that the infinitesimal (i.e., first order) deformations are classified by

$$H^2 = \operatorname{span}_{\mathbb{R}}(\upsilon^0\upsilon^+ S_+, \upsilon^0\upsilon^- S_-, \sigma^+\upsilon^- V_0, \sigma^-\upsilon^+ V_0, \sigma^+\upsilon^0 V_+, \sigma^-\upsilon^0 V_-). \tag{10}$$

The most general infinitesimal deformation is therefore given by

$$\mu_1 = t_1^+ \upsilon^0\upsilon^+ S_+ + t_1^- \upsilon^0\upsilon^- S_- + t_2^+ \sigma^+\upsilon^- V_0 + t_2^- \sigma^-\upsilon^+ V_0 + t_3^+ \sigma^+\upsilon^0 V_+ + t_3^- \sigma^-\upsilon^0 V_-, \tag{11}$$

where we have introduced parameters $t_1^\pm, t_2^\pm, t_3^\pm \in \mathbb{R}$.

2.3. **Obstructions.** The obstructions to integrating the infinitesimal deformation μ_1 are classes in H^3, the first of which is the class of $\frac{1}{2}[\![\mu_1, \mu_1]\!]$, where $[\![-,-]\!] : C^p \times C^q \to C^{p+q-1}$ is the Nijenhuis–Richardson bracket [11], which defines a graded Lie superalgebra structure on $A^\bullet = \bigoplus_{p \geqslant -1} A^p$, where $A^p = C^{p+1}$. If $\alpha \otimes X \in C^{p+1}$ and $\beta \otimes Y \in C^{q+1}$, then their Nijenhuis–Richardson bracket is given by a graded commutator

$$[\![\alpha \otimes X, \beta \otimes Y]\!] = (\alpha \otimes X) \bullet (\beta \otimes Y) - (-1)^{pq} (\beta \otimes Y) \bullet (\alpha \otimes X), \tag{12}$$

where the operation $\bullet$, which is *not* associative, is given by

$$(\alpha \otimes X) \bullet (\beta \otimes Y) = \alpha \wedge \imath_X \beta \otimes Y. \tag{13}$$

Table 1 lists the $\bullet$ product $\bullet : C^2 \times C^2 \to C^3$, from where we can read off the Nijenhuis–Richardson brackets. The shorthand notation in that table is such that, for example,

$$\upsilon^0\upsilon^+\upsilon^- V_0 := \upsilon_a^0 \wedge \upsilon_a^+ \wedge \upsilon_b^- \otimes V_{0\,b}, \tag{14}$$

et cetera.

TABLE 1. Some components of the Nijenhuis–Richardson product $\bullet : C^2 \times C^2 \to C^3$

$\bullet$	$\upsilon^0\upsilon^+ S_+$	$\upsilon^0\upsilon^- S_-$	$\sigma^+\upsilon^- V_0$	$\sigma^-\upsilon^+ V_0$	$\sigma^+\upsilon^0 V_+$	$\sigma^-\upsilon^0 V_-$
$\upsilon^0\upsilon^+ S_+$			$\upsilon^0\upsilon^+\upsilon^- V_0$		$\upsilon^0\upsilon^+\upsilon^0 V_+$	
$\upsilon^0\upsilon^- S_-$				$\upsilon^0\upsilon^-\upsilon^+ V_0$		$\upsilon^0\upsilon^-\upsilon^0 V_-$
$\sigma^+\upsilon^- V_0$	$-\sigma^+\upsilon^+\upsilon^- S_+$					$\sigma^+\sigma^-\upsilon^- V_-$
$\sigma^-\upsilon^+ V_0$		$\sigma^-\upsilon^+\upsilon^- S_-$			$-\sigma^+\sigma^-\upsilon^+ V_+$	
$\sigma^+\upsilon^0 V_+$				$\sigma^+\sigma^-\upsilon^0 V_0$		
$\sigma^-\upsilon^0 V_-$			$-\sigma^+\sigma^-\upsilon^0 V_0$			

Calculating $\frac{1}{2}[\![\mu_1,\mu_1]\!] = \mu_1 \bullet \mu_1$, we find

$$\begin{split}\tfrac{1}{2}[\![\mu_1,\mu_1]\!] = t_1^+t_2^+\upsilon^0\upsilon^+\upsilon^-V_0 + t_1^+t_3^+\upsilon^0\upsilon^+\upsilon^0V_+ + t_1^-t_2^-\upsilon^0\upsilon^-\upsilon^+V_0 + t_1^-t_3^-\upsilon^0\upsilon^-\upsilon^0V_- - t_1^+t_2^+\sigma^+\upsilon^+\upsilon^-S_+\\ + t_1^-t_2^-\sigma^-\upsilon^+\upsilon^-S_- - t_2^-t_3^+\sigma^+\sigma^-\upsilon^+V_+ + t_2^+t_3^-\sigma^+\sigma^-\upsilon^-V_- + (t_2^-t_3^+ - t_2^+t_3^-)\sigma^+\sigma^-\upsilon^0V_0. \quad (15)\end{split}$$

This has to be a coboundary, so equal to $\partial\mu_2$, where

$$\mu_2 = u_1\sigma^+\sigma^-D + u_2\upsilon^+\upsilon^-D + u_3\upsilon^+\upsilon^-J + \tfrac{1}{2}u_4\upsilon^0\upsilon^0J, \quad (16)$$

for some $u_1,u_2,u_3,u_4 \in \mathbb{R}$. From equation (9), we see that

$$\begin{split}\partial\mu_2 = u_3\upsilon^0\upsilon^+\upsilon^-V_0 - u_4\upsilon^0\upsilon^+\upsilon^0V_+ + u_3\upsilon^0\upsilon^-\upsilon^+V_0 - u_4\upsilon^0\upsilon^-\upsilon^0V_- - u_2\sigma^+\upsilon^+\upsilon^-S_+ + u_2\sigma^-\upsilon^+\upsilon^-S_-\\ - u_1\sigma^+\sigma^-\upsilon^+V_+ + u_1\sigma^+\sigma^-\upsilon^-V_- + (u_2-u_3)(-\upsilon^+\upsilon^-\upsilon^+V_+ + \upsilon^+\upsilon^-\upsilon^-V_-) \quad (17)\end{split}$$

The first obstruction equation $\partial\mu_2 = \frac{1}{2}[\![\mu_1,\mu_1]\!]$ has a solution provided that the following equations are satisfied:

$$\begin{aligned} u_1 &= t_2^+t_3^- & \qquad & t_1^+t_2^+ = t_1^-t_2^-\\ u_2 = u_3 &= t_1^+t_2^+ & \text{and} \qquad & t_1^+t_3^+ = t_1^-t_3^- \\ u_4 &= -t_1^+t_3^+ & & t_2^+t_3^- = t_2^-t_3^+,\end{aligned} \quad (18)$$

in which case the infinitesimal deformation μ_1 integrates to second order with quadratic terms

$$\mu_2 = t_2^+t_3^-\sigma^+\sigma^-D + t_1^+t_2^+(\upsilon^+\upsilon^-D + \upsilon^+\upsilon^-J) - \tfrac{1}{2}t_1^+t_3^+\upsilon^0\upsilon^0J. \quad (19)$$

TABLE 2. More components of the Nijenhuis–Richardson product $\bullet : C^2 \times C^2 \to C^3$

$\bullet$	$\sigma^+\sigma^-D$	$\upsilon^+\upsilon^-D$	$\upsilon^+\upsilon^-J$	$\frac{1}{2}\upsilon^0\upsilon^0J$
$\upsilon^0\upsilon^+S_+$	$\sigma^-\upsilon^0\upsilon^+D$			
$\upsilon^0\upsilon^-S_-$	$-\sigma^+\upsilon^0\upsilon^-D$			
$\sigma^+\upsilon^-V_0$				$\sigma^+\upsilon^0\upsilon^-J$
$\sigma^-\upsilon^+V_0$				$\sigma^-\upsilon^0\upsilon^+J$
$\sigma^+\upsilon^0V_+$		$\sigma^+\upsilon^0\upsilon^-D$	$\sigma^+\upsilon^0\upsilon^-J$	
$\sigma^-\upsilon^0V_-$		$-\sigma^-\upsilon^0\upsilon^+D$	$\sigma^-\upsilon^0\upsilon^+J$	

To probe integrability to third order, we must solve $\partial\mu_3 = [\![\mu_1,\mu_2]\!]$ for some μ_3. Since there are no cochains with legs on D or J_{ab} in our complex, $\mu_2 \bullet \mu_1 = 0$ and hence $[\![\mu_1,\mu_2]\!] = \mu_1 \bullet \mu_2$, which can be calculated making use of Table 2. Using equation (18), one finds that $\mu_1 \bullet \mu_2 = 0$, so we can take $\mu_3 = 0$. We then notice that $[\![\mu_2,\mu_2]\!] = 0$ identically, so that $\mu_4 = 0$ and hence all higher deformations vanish.

In summary, the most general integrable deformation is given by

$$\begin{split}\mu = t_1^+\upsilon^0\upsilon^+S_+ + t_1^-\upsilon^0\upsilon^-S_- + t_2^+\sigma^+\upsilon^-V_0 + t_2^-\sigma^-\upsilon^+V_0 + t_3^+\sigma^+\upsilon^0V_+ + t_3^-\sigma^-\upsilon^0V_-\\ + t_2^+t_3^-\sigma^+\sigma^-D + t_1^+t_2^+(\upsilon^+\upsilon^-D + \upsilon^+\upsilon^-J) - \tfrac{1}{2}t_1^+t_3^+\upsilon^0\upsilon^0J, \quad (20)\end{split}$$

subject to the following quadratic equations:

$$t_1^+t_2^+ = t_1^-t_2^-, \qquad t_1^+t_3^+ = t_1^-t_3^- \qquad \text{and} \qquad t_2^+t_3^- = t_2^-t_3^+. \quad (21)$$

These equations are easy to interpret: they simply state that (t_1^-,t_1^+), (t_2^+,t_2^-) and (t_3^+,t_3^-) are collinear vectors in $\mathbb{R}^2$. In other words, there exists a *nonzero* vector $(x,y) \in \mathbb{R}^2$ and reals $\alpha_1,\alpha_2,\alpha_3$ so that

$$(t_1^-,t_1^+) = \alpha_1(x,y), \qquad (t_2^+,t_2^-) = \alpha_2(x,y) \qquad \text{and} \qquad (t_3^+,t_3^-) = \alpha_3(x,y). \quad (22)$$

Then the deformation becomes

$$\begin{split}\mu = \alpha_1 y\upsilon^0\upsilon^+S_+ + \alpha_1 x\upsilon^0\upsilon^-S_- + \alpha_2 x\sigma^+\upsilon^-V_0 + \alpha_2 y\sigma^-\upsilon^+V_0 + \alpha_3 x\sigma^+\upsilon^0V_+ + \alpha_3 y\sigma^-\upsilon^0V_-\\ + \alpha_2\alpha_3 xy\sigma^+\sigma^-D + \alpha_1\alpha_2 xy(\upsilon^+\upsilon^-D + \upsilon^+\upsilon^-J) - \tfrac{1}{2}\alpha_1\alpha_3 xy\upsilon^0\upsilon^0J. \quad (23)\end{split}$$

This results in the following Lie brackets in addition to those in (3):

$$\begin{aligned} [S_+,V_{0\,a}] &= \alpha_3 xV_{+\,a} & & [V_{0\,a},V_{0\,b}] = -\alpha_1\alpha_3 xyJ_{ab}\\ [S_-,V_{0\,a}] &= \alpha_3 yV_{-\,a} & & [V_{0\,a},V_{+\,b}] = \alpha_1 y\delta_{ab}S_+\\ [S_+,V_{-\,a}] &= \alpha_2 xV_{0\,a} & \text{and} \quad & [V_{0\,a},V_{-\,b}] = \alpha_1 x\delta_{ab}S_-\\ [S_-,V_{+\,a}] &= \alpha_2 yV_{0\,a} & & [V_{+\,a},V_{-\,b}] = \alpha_1\alpha_2 xy(J_{ab} + \delta_{ab}D).\\ [S_+,S_-] &= \alpha_2\alpha_3 xyD & & \end{aligned} \quad (24)$$

2.4. **Isomorphism classes of deformations.** Clearly the parameters in equation (24) are not effective. To bring these brackets to normal form, we need to identify Lie algebras which are related by automorphisms of the static graded conformal algebra, which is the "gauge" symmetry of the deformation complex. We will not need to determine the full automorphism group, but it will suffice to consider automorphisms of two types. One is a $\mathbb{Z}_2$ subgroup which acts by

$$J \mapsto J, \qquad D \mapsto -D, \qquad S_\pm \mapsto S_\mp, \qquad V_\pm \mapsto V_\mp \qquad \text{and} \qquad V_0 \mapsto V_0. \tag{25}$$

Its effect on the Lie brackets in (24) is to exchange x and y. The second type of automorphism acts by rescaling some of the generators:

$$J \mapsto J, \qquad D \mapsto D, \qquad S_\pm \mapsto \alpha_\pm S_\pm, \qquad V_\pm \mapsto \beta_\pm V_\pm \qquad \text{and} \qquad V_0 \mapsto \gamma V_0, \tag{26}$$

for some $\alpha_\pm, \beta_\pm, \gamma \in \mathbb{R}^\times$.

There are three cases to consider: $x = 0$, $y = 0$ and $xy \neq 0$. The first two are actually related by the automorphism (25), so this leaves two cases to consider: $y = 0$ (and hence $x \neq 0$) and $xy \neq 0$.

2.4.1. *Branch* $y = 0$ *(and hence* $x \neq 0$*).* In this case the nonzero brackets in addition to those in (3) are

$$\begin{aligned} [S_+, V_{0\,a}] &= \alpha_3 x V_{+\,a} \\ [S_+, V_{-\,a}] &= \alpha_2 x V_{0\,a} \\ [V_{0\,a}, V_{-\,b}] &= \alpha_1 x \delta_{ab} S_-. \end{aligned} \tag{27}$$

Via automorphism of the type (26), we can bring any nonzero $\alpha_i x$ to 1. Therefore we have eight isomorphism classes of graded conformal algebras: the static graded conformal algebra and seven nontrivial deformations, depending on the whether or not each of $\alpha_{1,2,3}$ vanishes. The isomorphism classes of such Lie algebras are tabulated in Table 3.

TABLE 3. Isomorphism classes of graded conformal algebras ($y = 0$)

α_1	α_2	α_3	Nonzero Lie brackets in addition to (3)		
0	0	0			
1	0	0	$[V_0, V_-] = S_-$		
0	1	0		$[S_+, V_-] = V_0$	
0	0	1			$[S_+, V_0] = V_+$
1	1	0	$[V_0, V_-] = S_-$	$[S_+, V_-] = V_0$	
1	0	1	$[V_0, V_-] = S_-$		$[S_+, V_0] = V_+$
0	1	1		$[S_+, V_-] = V_0$	$[S_+, V_0] = V_+$
1	1	1	$[V_0, V_-] = S_-$	$[S_+, V_-] = V_0$	$[S_+, V_0] = V_+$

2.4.2. *Branch* $xy \neq 0$*.* We can now use automorphisms of the type (26) in order to rescale the generators and bring the structure constants to a normal form. Under the automorphism (26), we find

$$\begin{aligned} \alpha_1 x &\mapsto \frac{\alpha_-}{\gamma\beta_-}\alpha_1 x \\ \alpha_2 x &\mapsto \frac{\gamma}{\alpha_+\beta_-}\alpha_2 x \\ \alpha_3 x &\mapsto \frac{\beta_+}{\gamma\alpha_+}\alpha_3 x \end{aligned} \qquad \text{and} \qquad \begin{aligned} \alpha_1 y &\mapsto \frac{\alpha_+}{\gamma\beta_+}\alpha_1 y \\ \alpha_2 y &\mapsto \frac{\gamma}{\alpha_-\beta_+}\alpha_2 y \\ \alpha_3 y &\mapsto \frac{\beta_-}{\gamma\alpha_-}\alpha_3 y. \end{aligned} \tag{28}$$

Table 4 shows to what normal form we can bring $\alpha_i x$ and $\alpha_i y$ depending on whether or not each α_i vanishes. The notation is such that ε is the sign of $\alpha_1\alpha_3 xy$, which is an invariant: indeed, under the above rescalings $\alpha_1\alpha_3 xy \mapsto \gamma^{-2}\alpha_1\alpha_3 xy$, so the sign cannot change..

The resulting isomorphism classes of Lie algebras are tabulated in Table 5, where we recognise the Lie algebras (4).

2.5. **Deformations for** $d = 4$**.** If $d = 4$ all that happens is that $\mathfrak{so}(4)$ is not simple and hence we can decompose the rotation generators J_{ab} into its (anti) self-dual parts $J^\pm_{ab}$. (The sign has nothing to do with the D-weight, despite the notation.) Following the above calculation now in $d = 4$ we see that the infinitesimal deformations do not change, so that μ_1 is formally as in $d \geqslant 5$ and hence so is $\frac{1}{2}[\![\mu_1, \mu_1]\!]$. The cochain μ_2 changes in principle, since now

$$\mu_2 = u_1\sigma^+\sigma^- D + u_2 v^+ v^- D + u_3^+ v^+ v^- J^+ + u_3^- v^+ v^- J^- + u_4^+ \tfrac{1}{2} v^0 v^0 J^+ + u_4^- \tfrac{1}{2} v^0 v^0 J^-. \tag{29}$$

TABLE 4. Normal forms for structure constants

α_1	α_2	α_3	$\alpha_1 x$	$\alpha_1 y$	$\alpha_2 x$	$\alpha_2 y$	$\alpha_3 x$	$\alpha_3 y$
0	0	0	0	0	0	0	0	0
1	0	0	1	1	0	0	0	0
0	1	0	0	0	1	1	0	0
0	0	1	0	0	0	0	1	1
1	1	0	1	1	1	1	0	0
1	0	1	ε	ε	0	0	1	1
0	1	1	0	0	1	1	1	1
1	1	1	ε	ε	1	1	1	1

TABLE 5. Isomorphism classes of graded conformal algebras ($xy \neq 0$)

α_1	α_2	α_3	Nonzero Lie brackets in addition to (3)					
0	0	0						
1	0	0					$[V_0, V_\pm] = S_\pm$	
0	1	0		$[S_\pm, V_\mp] = V_0$				
0	0	1			$[S_\pm, V_0] = V_\pm$			
1	1	0		$[S_\pm, V_\mp] = V_0$			$[V_0, V_\pm] = S_\pm$	$[V_+, V_-] = (J + D)$
1	0	1			$[S_\pm, V_0] = V_\pm$	$[V_0, V_0] = -\varepsilon J$	$[V_0, V_\pm] = \varepsilon S_\pm$	
0	1	1	$[S_+, S_-] = D$	$[S_\pm, V_\mp] = V_0$	$[S_\pm, V_0] = V_\pm$			
1	1	1	$[S_+, S_-] = D$	$[S_\pm, V_\mp] = V_0$	$[S_\pm, V_0] = V_\pm$	$[V_0, V_0] = -\varepsilon J$	$[V_0, V_\pm] = \varepsilon S_\pm$	$[V_+, V_-] = \varepsilon(J + D)$

However, in order to cancel $\frac{1}{2}[\![\mu_1, \mu_1]\!]$ we must take $u_3^+ = u_3^- = u_3$ and $u_4^+ = u_4^- = u_4$, where u_3, u_4 are as in the $d \geqslant 5$ calculation, and thus μ_2 is formally equal to the one for $d \geqslant 5$. The rest of the calculation is formally identical to the $d \geqslant 5$ case, with identical results.

2.6. **Summary.** We can now summarise our results and list the isomorphism classes of graded conformal algebras with $d \geqslant 4$. These are displayed in Table 6. The first column is our label in this paper and should not be taken too seriously. The Lie algebra $\mathsf{GCA}_{15}^{(\varepsilon)}$ is isomorphic to $\mathfrak{so}(d+1, 2)$ if $\varepsilon = -1$ and to $\mathfrak{so}(d+2, 1)$ if $\varepsilon = 1$.

TABLE 6. Isomorphism classes of graded conformal algebras ($d \geqslant 4$)

Label	Nonzero Lie brackets in addition to (3)					
GCA_1						
GCA_2					$[V_0, V_-] = S_-$	
GCA_3		$[S_+, V_-] = V_0$				
GCA_4			$[S_+, V_0] = V_+$			
GCA_5		$[S_+, V_-] = V_0$			$[V_0, V_-] = S_-$	
GCA_6			$[S_+, V_0] = V_+$		$[V_0, V_-] = S_-$	
GCA_7		$[S_+, V_-] = V_0$	$[S_+, V_0] = V_+$			
GCA_8		$[S_+, V_-] = V_0$	$[S_+, V_0] = V_+$		$[V_0, V_-] = S_-$	
GCA_9					$[V_0, V_\pm] = S_\pm$	
GCA_{10}		$[S_\pm, V_\mp] = V_0$				
GCA_{11}			$[S_\pm, V_0] = V_\pm$			
GCA_{12}		$[S_\pm, V_\mp] = V_0$			$[V_0, V_\pm] = S_\pm$	$[V_+, V_-] = J + D$
$\mathsf{GCA}_{13}^{(\varepsilon=\pm 1)}$			$[S_\pm, V_0] = V_\pm$	$[V_0, V_0] = -\varepsilon J$	$[V_0, V_\pm] = \varepsilon S_\pm$	
GCA_{14}	$[S_+, S_-] = D$	$[S_\pm, V_\mp] = V_0$	$[S_\pm, V_0] = V_\pm$			
$\mathsf{GCA}_{15}^{(\varepsilon=\pm 1)}$	$[S_+, S_-] = D$	$[S_\pm, V_\mp] = V_0$	$[S_\pm, V_0] = V_\pm$	$[V_0, V_0] = -\varepsilon J$	$[V_0, V_\pm] = \varepsilon S_\pm$	$[V_+, V_-] = \varepsilon(J + D)$

2.7. **Invariant inner products.** Let us now consider whether any of the Lie algebras in Table 6 admit an invariant inner product; that is, a non-degenerate symmetric bilinear form $\langle -, - \rangle$ which is "associative" in the sense that

$$\langle [X,Y],Z\rangle = \langle X,[Y,Z]\rangle \tag{30}$$

for all X, Y, Z in the Lie algebra. As we will now see, with the exception of the simple Lie algebras $\mathsf{GCA}_{15}^{(\varepsilon)}$, none of the other Lie algebras in Table 6 admit an invariant inner product.

Indeed, invariance under D and J already says that for a graded conformal algebra the only possible nonzero components of such an inner product are

$$\langle S_+, S_-\rangle\,, \qquad \langle V_+, V_-\rangle\,, \qquad \langle V_0, V_0\rangle\,, \qquad \langle D, D\rangle \qquad \text{and} \qquad \langle J, J\rangle\,. \tag{31}$$

Concentrating on the first component and using (30), we find

$$\langle S_+, S_-\rangle = \langle [D, S_+], S_-\rangle = \langle D, [S_+, S_-]\rangle\,, \tag{32}$$

so that unless a D appears in $[S_+, S_-]$, the inner product $\langle S_+, S_-\rangle$ is zero and hence it is degenerate. Similarly,

$$\langle V_+, V_-\rangle = \langle [D, V_+], V_-\rangle = \langle D, [V_+, V_-]\rangle\,, \tag{33}$$

so that again the inner product is degenerate unless D appears in $[V_+, V_-]$. Inspecting Table 6 we see that only $\mathsf{GCA}_{15}^{(\varepsilon)}$ satisfies these conditions. Since these Lie algebras are simple, the Killing form is non-degenerate and hence they are the only metric Lie algebras in Table 6.

2.8. **Contractions.** It is clear from the expression (24) that every graded conformal algebra in $d \geqslant 4$ can be obtained as a contraction of $\mathsf{GCA}_{15}^{(\varepsilon)}$. Indeed, let $t \in (0,1]$ and consider the following invertible linear transformation on the underlying vector space of the Lie algebra:

$$\varphi_t J = J, \qquad \varphi_t D = D, \qquad \varphi_t S_\pm = t^{a_\pm} S_\pm, \qquad \varphi_t V_\pm = t^{b_\pm} V_\pm \qquad \text{and} \qquad \varphi_t V_0 = t^c V_0\,, \tag{34}$$

for some real numbers $a_\pm, b_\pm, c$. We define new Lie brackets (which are isomorphic for all $t \in (0,1]$)

$$[X,Y]_t = \varphi_t^{-1}[\varphi_t X, \varphi_t Y]. \tag{35}$$

If the limit $t \to 0$ of $[-,-]_t$ exists, then it defines a Lie algebra which is typically not isomorphic to the original Lie algebra at $t = 1$. The Lie algebra defined by $[-,-]_0$ is a contraction of the Lie algebra defined by $[-,-]_1$. If we start at $t = 1$ with $\mathsf{GCA}_{15}^{(\varepsilon)}$, then by choosing the weights $a_\pm, b_\pm, c$ judiciously we can arrive at all the other graded conformal algebras GCA_1 to GCA_{14}. Table 7 gives (non-unique) values for these weights for each of the non-simple graded conformal algebras. Of course, contracting with zero weights does not change the isomorphism class of the Lie algebra.

Table 7. Weights for contractions of simple conformal algebras

Contraction	a_+	a_-	b_+	b_-	c
$\mathsf{GCA}_{15} \to \mathsf{GCA}_1$	1	1	1	1	1
$\mathsf{GCA}_{15} \to \mathsf{GCA}_2$	1	2	1	1	1
$\mathsf{GCA}_{15} \to \mathsf{GCA}_3$	3	2	2	0	3
$\mathsf{GCA}_{15} \to \mathsf{GCA}_4$	1	1	2	1	1
$\mathsf{GCA}_{15} \to \mathsf{GCA}_5$	1	1	1	0	1
$\mathsf{GCA}_{15} \to \mathsf{GCA}_6$	1	2	2	1	1
$\mathsf{GCA}_{15} \to \mathsf{GCA}_7$	1	1	3	1	2

Contraction	a_+	a_-	b_+	b_-	c
$\mathsf{GCA}_{15} \to \mathsf{GCA}_8$	1	1	2	0	1
$\mathsf{GCA}_{15} \to \mathsf{GCA}_9$	2	1	1	0	1
$\mathsf{GCA}_{15} \to \mathsf{GCA}_{10}$	1	0	1	0	1
$\mathsf{GCA}_{15} \to \mathsf{GCA}_{11}$	1	0	2	1	1
$\mathsf{GCA}_{15} \to \mathsf{GCA}_{12}$	1	1	0	0	1
$\mathsf{GCA}_{15} \to \mathsf{GCA}_{13}$	1	0	1	0	0
$\mathsf{GCA}_{15} \to \mathsf{GCA}_{14}$	0	0	1	1	1

Contractions define a partial order in the set of isomorphism classes of conformal Lie algebras and, as any partial order, they have an associated Hasse diagram. This diagram appears in Figure 1, which also takes into account the graded conformal algebras for $d = 3$ and $d = 2$. The red vertices depict the algebras unique to $d = 3$. Deleting these vertices and any edges from it, one recovers the Hasse diagram for graded conformal algebras in $d \geqslant 4$ and, as we shall see below, also for $d = 2$.

3. Deformations for $d = 3$

We now extend the above classification to $d = 3$.

3.1. **The deformation complex.** The existence of the Levi-Civita symbol ϵ_{abc} now embiggens the deformation complex. There are now additional cochains in C^1 and C^2:

$$\begin{aligned} C^1 &= C^1_{d\geqslant 4} \oplus \operatorname{span}_{\mathbb{R}}(\tfrac{1}{2}\upsilon^0 J) \\ C^2 &= C^2_{d\geqslant 4} \oplus \operatorname{span}_{\mathbb{R}}(\upsilon^+\upsilon^- V_0, \upsilon^0\upsilon^+ V_+, \upsilon^0\upsilon^- V_-, \tfrac{1}{2}\upsilon^0\upsilon^0 V_0, \tfrac{1}{2}\sigma^+\upsilon^- J, \tfrac{1}{2}\sigma^-\upsilon^+ J), \end{aligned} \tag{36}$$

where $C^{1,2}_{d\geqslant 4}$ are given by equation (7) and where we continue using a shorthand notation where, for example,

$$\upsilon^0 J := \epsilon_{abc}\upsilon^0_a \otimes J_{bc}, \qquad \upsilon^+\upsilon^- V_0 := \epsilon_{abc}\upsilon^+_a \wedge \upsilon^-_b \otimes V_{0\,c}, \tag{37}$$

et cetera.

3.2. Infinitesimal deformations. The differential on generators is unchanged, but now $\partial : C^1 \to C^2$ is no longer the zero map, since

$$\partial(\tfrac{1}{2}\upsilon^0 J) = -\upsilon^0\upsilon^+ V_+ - \upsilon^0\upsilon^- V_- - \upsilon^0\upsilon^0 V_0, \tag{38}$$

whose right-hand side spans the space B^2 of 2-coboundaries. The 2-cocycles are now

$$Z^2 = \operatorname{span}_{\mathbb{R}}(\upsilon^0\upsilon^+S_+, \upsilon^0\upsilon^-S_-, \sigma^+\upsilon^-V_0, \sigma^-\upsilon^+V_0, \sigma^+\upsilon^0V_+, \sigma^-\upsilon^0V_-, \upsilon^+\upsilon^-V_0, \upsilon^0\upsilon^+V_+, \upsilon^0\upsilon^-V_-, \tfrac{1}{2}\upsilon^0\upsilon^0V_0). \tag{39}$$

We may use the freedom to modify cocycles by coboundaries in order to "normalise" the cocycles and in this way choose a unique cocycle representative for each cohomology class. A convenient normalisation condition is to demand that the coefficient of $\frac{1}{2}\upsilon^0\upsilon^0V_0$ should be zero. This can always be achieved by adding a suitable multiple of $\partial(\frac{1}{2}\upsilon^0 J)$. It is clear that every cohomology class has a unique normalised cocycle and hence, in summary,

$$H^2 \cong \operatorname{span}_{\mathbb{R}}(\upsilon^0\upsilon^+S_+, \upsilon^0\upsilon^-S_-, \sigma^+\upsilon^-V_0, \sigma^-\upsilon^+V_0, \sigma^+\upsilon^0V_+, \sigma^-\upsilon^0V_-, \upsilon^+\upsilon^-V_0, \upsilon^0\upsilon^+V_+, \upsilon^0\upsilon^-V_-). \tag{40}$$

The most general infinitesimal deformation is then given by

$$\begin{split}\mu_1 = t_1^+\upsilon^0\upsilon^+S_+ + t_1^-\upsilon^0\upsilon^-S_- + t_2^+\sigma^+\upsilon^-V_0 + t_2^-\sigma^-\upsilon^+V_0 + t_3^+\sigma^+\upsilon^0V_+ + t_3^-\sigma^-\upsilon^0V_-\\ + t_4\upsilon^+\upsilon^-V_0 + t_5^+\upsilon^0\upsilon^+V_+ + t_5^-\upsilon^0\upsilon^-V_-,\end{split} \tag{41}$$

for some real parameters $t_1^\pm, t_2^\pm, t_3^\pm, t_4, t_5^\pm$.

3.3. Obstructions. The first obstruction to integrating μ_1 is given by $\frac{1}{2}[\![\mu_1,\mu_1]\!]$. We can reuse some of the calculations in Section 2.3 and embed Table 1 into Table 8.

TABLE 8. Some components of $\bullet : C^2 \times C^2 \to C^3$

$\bullet$	$\upsilon^0\upsilon^+S_+$	$\upsilon^0\upsilon^-S_-$	$\sigma^+\upsilon^-V_0$	$\sigma^-\upsilon^+V_0$	$\sigma^+\upsilon^0V_+$	$\sigma^-\upsilon^0V_-$	$\upsilon^+\upsilon^-V_0$	$\upsilon^0\upsilon^+V_+$	$\upsilon^0\upsilon^-V_-$
$\upsilon^0\upsilon^+S_+$			$\upsilon^0\upsilon^+\upsilon^-V_0$		$\upsilon^0\upsilon^+\upsilon^0V_+$				
$\upsilon^0\upsilon^-S_-$				$\upsilon^0\upsilon^-\upsilon^+V_0$		$\upsilon^0\upsilon^-\upsilon^0V_-$			
$\sigma^+\upsilon^-V_0$	$-\sigma^+\upsilon^+\upsilon^-S_+$					$\sigma^+\sigma^-\upsilon^-V_-$		$\sigma^+\upsilon^+\upsilon^-V_+$	$\sigma^+\upsilon^-\upsilon^-V_-$
$\sigma^-\upsilon^+V_0$		$\sigma^-\upsilon^+\upsilon^-S_-$			$-\sigma^+\sigma^-\upsilon^+V_+$			$\sigma^-\upsilon^+\upsilon^+V_+$	$\sigma^-\upsilon^+\upsilon^-V_-$
$\sigma^+\upsilon^0V_+$				$\sigma^+\sigma^-\upsilon^0V_0$			$\sigma^+\upsilon^0\upsilon^-V_0$	$\sigma^+\upsilon^0\upsilon^0V_+$	
$\sigma^-\upsilon^0V_-$			$-\sigma^+\sigma^-\upsilon^0V_0$				$\sigma^-\upsilon^0\upsilon^+V_0$		$\sigma^-\upsilon^0\upsilon^0V_-$
$\upsilon^+\upsilon^-V_0$	$\upsilon^+\upsilon^+\upsilon^-S_+$	$\upsilon^+\upsilon^-\upsilon^-S_-$			$-\sigma^+\upsilon^+\upsilon^-V_+$	$-\sigma^-\upsilon^+\upsilon^-V_-$		$\upsilon^+\upsilon^-\upsilon^+V_+$	$-\upsilon^+\upsilon^-\upsilon^-V_-$
$\upsilon^0\upsilon^+V_+$	$-\upsilon^0\upsilon^0\upsilon^+S_+$			$-\sigma^-\upsilon^0\upsilon^+V_0$			$\upsilon^+\upsilon^-\upsilon^0V_0$ $-\upsilon^0\upsilon^-\upsilon^+V_0$	$\upsilon^0\upsilon^+\upsilon^0V_+$	
$\upsilon^0\upsilon^-V_-$		$-\upsilon^0\upsilon^0\upsilon^-S_-$	$-\sigma^+\upsilon^0\upsilon^-V_0$				$\upsilon^+\upsilon^-\upsilon^0V_0$ $-\upsilon^0\upsilon^+\upsilon^-V_0$		$\upsilon^0\upsilon^-\upsilon^0V_-$

We find that $\frac{1}{2}[\![\mu_1,\mu_1]\!] = \mu_1 \bullet \mu_1$ is given by

$$\begin{split}\tfrac{1}{2}[\![\mu_1,\mu_1]\!] &= (t_1^+t_2^+ - t_4t_5^-)\upsilon^0\upsilon^+\upsilon^-V_0 + (t_1^+t_3^+ + (t_5^+)^2)\upsilon^0\upsilon^+\upsilon^0V_+ + (t_1^-t_2^- - t_4t_5^+)\upsilon^0\upsilon^-\upsilon^+V_0\\ &+ (t_1^-t_3^- + (t_5^-)^2)\upsilon^0\upsilon^-\upsilon^0V_- - t_1^+t_2^+\sigma^+\upsilon^+\upsilon^-S_+ + t_1^-t_2^-\sigma^-\upsilon^+\upsilon^-S_- - t_2^-t_3^+\sigma^+\sigma^-\upsilon^+V_+\\ &+ t_2^+t_3^-\sigma^+\sigma^-\upsilon^-V_- + (t_2^-t_3^+ - t_2^+t_3^-)\sigma^+\sigma^-\upsilon^0V_0 + t_4(t_5^+ + t_5^-)\upsilon^+\upsilon^-\upsilon^0V_0 + t_4t_5^+\upsilon^+\upsilon^-\upsilon^+V_+\\ &- t_4t_5^-\upsilon^+\upsilon^-\upsilon^-V_- + t_1^+t_4\upsilon^+\upsilon^+\upsilon^-S_+ + t_1^-t_4\upsilon^+\upsilon^-\upsilon^-S_- - t_1^+t_5^+\upsilon^0\upsilon^0\upsilon^+S_+ - t_1^-t_5^-\upsilon^0\upsilon^0\upsilon^-S_-\\ &+ (t_3^+t_4 - t_2^+t_5^-)\sigma^+\upsilon^0\upsilon^-V_0 + t_3^+t_5^+\sigma^+\upsilon^0\upsilon^0V_+ + t_2^+t_5^-\sigma^+\upsilon^-\upsilon^-V_- + (t_2^+t_5^+ - t_3^+t_4)\sigma^+\upsilon^+\upsilon^-V_+\\ &+ (t_3^-t_4 - t_2^-t_5^+)\sigma^-\upsilon^0\upsilon^+V_0 + t_3^-t_5^-\sigma^-\upsilon^0\upsilon^0V_- + t_2^-t_5^+\sigma^-\upsilon^+\upsilon^+V_+ + (t_2^-t_5^- - t_3^-t_4)\sigma^-\upsilon^+\upsilon^-V_-,\end{split} \tag{42}$$

which has to equal $\partial\mu_2$ with

$$\mu_2 = u_1\sigma^+\sigma^-D + u_2\upsilon^+\upsilon^-D + u_3\upsilon^+\upsilon^-J + \tfrac{1}{2}u_4\upsilon^0\upsilon^0J + \tfrac{1}{2}u_5^+\sigma^+\upsilon^-J + \tfrac{1}{2}u_5^-\sigma^-\upsilon^+J. \tag{43}$$

We calculate

$$\begin{split}\partial\mu_2 &= u_3\upsilon^0\upsilon^+\upsilon^-V_0 - u_4\upsilon^0\upsilon^+\upsilon^0V_+ + u_3\upsilon^0\upsilon^-\upsilon^+V_0 - u_4\upsilon^0\upsilon^-\upsilon^0V_- - u_2\sigma^+\upsilon^+\upsilon^-S_+ + u_2\sigma^-\upsilon^+\upsilon^-S_-\\ &- u_1\sigma^+\sigma^-\upsilon^+V_+ + u_1\sigma^+\sigma^-\upsilon^-V_- + (u_2 - u_3)(-\upsilon^+\upsilon^-\upsilon^+V_+ + \upsilon^+\upsilon^-\upsilon^-V_-)\\ &+ u_5^+(\sigma^+\upsilon^+\upsilon^-V_+ + \sigma^+\upsilon^-\upsilon^-V_- + \sigma^+\upsilon^0\upsilon^-V_0) + u_5^-(\sigma^-\upsilon^+\upsilon^+V_+ + \sigma^-\upsilon^+\upsilon^-V_- + \sigma^-\upsilon^0\upsilon^+V_0).\end{split} \tag{44}$$

The equation $\partial\mu_2 = \frac{1}{2}[\![\mu_1,\mu_1]\!]$ is equivalent to the following conditions:

$$\begin{aligned} u_1 &= t_2^+t_3^- = t_2^-t_3^+ \\ u_2 = u_3 &= t_1^+t_2^+ = t_1^-t_2^- \\ u_4 &= -t_1^+t_3^+ - (t_5^+)^2 = -t_1^-t_3^- - (t_5^-)^2 \\ u_5^\pm &= t_2^\pm t_5^\mp = t_2^\pm t_5^\pm - t_3^\pm t_4 = t_3^\pm t_4 - t_2^\pm t_5^\mp \end{aligned} \qquad \text{and} \qquad \begin{aligned} t_1^\pm t_4 &= 0 \\ t_1^\pm t_5^\pm &= 0 \\ t_3^\pm t_5^\pm &= 0 \\ t_4 t_5^\pm &= 0, \end{aligned} \tag{45}$$

after a little simplification.

Table 9. More components of $\bullet : C^2 \times C^2 \to C^3$

$\bullet$	$\sigma^+\sigma^-D$	$\upsilon^+\upsilon^-D$	$\upsilon^+\upsilon^-J$	$\frac{1}{2}\upsilon^0\upsilon^0J$	$\frac{1}{2}\sigma^+\upsilon^-J$	$\frac{1}{2}\sigma^-\upsilon^+J$
$\upsilon^0\upsilon^+S_+$	$\sigma^-\upsilon^0\upsilon^+D$				$\frac{1}{2}\upsilon^0\upsilon^+\upsilon^-J$	
$\upsilon^0\upsilon^-S_-$	$-\sigma^+\upsilon^0\upsilon^-D$					$\frac{1}{2}\upsilon^0\upsilon^-\upsilon^+J$
$\sigma^+\upsilon^-V_0$				$\sigma^+\upsilon^0\upsilon^-J$		
$\sigma^-\upsilon^+V_0$				$\sigma^-\upsilon^0\upsilon^+J$		
$\sigma^+\upsilon^0V_+$		$\sigma^+\upsilon^0\upsilon^-D$	$\sigma^+\upsilon^0\upsilon^-J$			$\frac{1}{2}\sigma^+\sigma^-\upsilon^0J$
$\sigma^-\upsilon^0V_-$		$-\sigma^-\upsilon^0\upsilon^+D$	$\sigma^-\upsilon^0\upsilon^+J$		$-\frac{1}{2}\sigma^+\sigma^-\upsilon^0J$	
$\upsilon^+\upsilon^-V_0$				$\frac{1}{2}\upsilon^0\upsilon^+\upsilon^-J$ $+\frac{1}{2}\upsilon^0\upsilon^-\upsilon^+J$		
$\upsilon^0\upsilon^+V_+$		$\upsilon^0\upsilon^+\upsilon^-D$	$-\frac{1}{2}\upsilon^0\upsilon^-\upsilon^+J$ $-\frac{1}{2}\upsilon^+\upsilon^-\upsilon^0J$			$-\sigma^-\upsilon^0\upsilon^+J$
$\upsilon^0\upsilon^-V_-$		$-\upsilon^0\upsilon^+\upsilon^-D$	$\frac{1}{2}\upsilon^+\upsilon^-\upsilon^0J$ $-\frac{1}{2}\upsilon^0\upsilon^+\upsilon^-J$		$-\sigma^+\upsilon^0\upsilon^-J$	

To determine the obstruction to integrability to third order we need to calculate $[\![\mu_1,\mu_2]\!]$, for which we extend Table 2 to the larger Table 9 which includes the new cochains. In arriving at some of the new entries we have used the identity

$$J_{ab} = \tfrac{1}{2}\epsilon_{abc}\epsilon_{cde}J_{de}. \tag{46}$$

As before, $[\![\mu_1,\mu_2]\!] = \mu_1 \bullet \mu_2$, since $\mu_2 \bullet \mu_1 = 0$. We calculate

$$\begin{aligned}[\![\mu_1,\mu_2]\!] = (t_1^+u_1 - t_3^-u_2)\sigma^-\upsilon^0\upsilon^+D + (t_3^+u_2 - t_1^-u_1)\sigma^+\upsilon^0\upsilon^-D + (t_5^+ - t_5^-)u_2\upsilon^0\upsilon^+\upsilon^-D \\ + (t_2^+u_4 + t_3^+u_3 - t_5^-u_5^+)\sigma^+\upsilon^0\upsilon^-J + (t_2^-u_4 + t_3^-u_3 - t_5^+u_5^-)\sigma^-\upsilon^0\upsilon^+J \\ + \tfrac{1}{2}(t_4u_4 - t_5^-u_3 + t_1^+u_5^+)\upsilon^0\upsilon^+\upsilon^-J + \tfrac{1}{2}(t_4u_4 - t_5^+u_3 + t_1^-u_5^-)\upsilon^0\upsilon^-\upsilon^+J \\ + \tfrac{1}{2}(t_5^- - t_5^+)u_3\upsilon^+\upsilon^-\upsilon^0J + \tfrac{1}{2}(t_3^+u_5^- - t_3^-u_5^+)\sigma^+\sigma^-\upsilon^0J,\end{aligned} \tag{47}$$

which, after using conditions (45), reduces to

$$[\![\mu_1,\mu_2]\!] = -t_2^+\left((t_5^+)^2 + (t_5^-)^2\right)\sigma^+\upsilon^0\upsilon^-J - t_2^-\left((t_5^+)^2 + (t_5^-)^2\right)\sigma^-\upsilon^0\upsilon^+J. \tag{48}$$

This cannot be cancelled with $\partial\mu_3$ unless it actually vanishes, which implies the extra conditions

$$t_2^\pm\left((t_5^+)^2 + (t_5^-)^2\right) = 0. \tag{49}$$

This being the case, the obstruction is overcome with $\mu_3 = 0$ and since $[\![\mu_2,\mu_2]\!]$ is identically zero, also all higher μ_i vanish. In summary, the most general integrable deformation is given by

$$\begin{aligned}\mu = t_1^+\upsilon^0\upsilon^+S_+ + t_1^-\upsilon^0\upsilon^-S_- + t_2^+\sigma^+\upsilon^-V_0 + t_2^-\sigma^-\upsilon^+V_0 + t_3^+\sigma^+\upsilon^0V_+ + t_3^-\sigma^-\upsilon^0V_- + t_4\upsilon^+\upsilon^-V_0 \\ + t_5^+\upsilon^0\upsilon^+V_+ + t_5^-\upsilon^0\upsilon^-V_- + t_2^+t_3^-\sigma^+\sigma^-D + t_1^+t_2^+(\upsilon^+\upsilon^-D + \upsilon^+\upsilon^-J) \\ - \tfrac{1}{2}\left(t_1^+t_3^+ + (t_5^+)^2\right)\upsilon^0\upsilon^0J + \tfrac{1}{2}t_2^+t_5^-\sigma^+\upsilon^-J + \tfrac{1}{2}t_2^-t_5^+\sigma^-\upsilon^+J,\end{aligned} \tag{50}$$

subject to the conditions

$$\begin{aligned} t_2^+t_3^- &= t_2^-t_3^+ & t_2^\pm t_5^\pm &= 3t_2^\pm t_5^\mp \\ t_1^+t_2^+ &= t_1^-t_2^- & t_1^\pm t_4 &= 0 \\ t_1^+t_3^+ - t_1^-t_3^- &= (t_5^- - t_5^+)(t_5^- + t_5^+) & t_1^\pm t_5^\pm &= 0 \\ t_2^\pm\left((t_5^+)^2 + (t_5^-)^2\right) &= 0 & t_3^\pm t_5^\pm &= 0 \\ t_3^\pm t_4 &= 2t_2^\pm t_5^\mp & t_4 t_5^\pm &= 0. \end{aligned} \tag{51}$$

It follows from these conditions that

$$t_2^\pm t_5^\pm = 0, \qquad t_2^\pm t_5^\mp = 0 \qquad \text{and} \qquad t_3^\pm t_4 = 0, \tag{52}$$

and hence that $u_5^\pm = 0$. We may therefore summarise again the discussion by saying that the most general integrable deformation is

$$\begin{aligned}\mu = t_1^+\upsilon^0\upsilon^+S_+ + t_1^-\upsilon^0\upsilon^-S_- + t_2^+\sigma^+\upsilon^-V_0 + t_2^-\sigma^-\upsilon^+V_0 + t_3^+\sigma^+\upsilon^0V_+ + t_3^-\sigma^-\upsilon^0V_- + t_4\upsilon^+\upsilon^-V_0 \\ + t_5^+\upsilon^0\upsilon^+V_+ + t_5^-\upsilon^0\upsilon^-V_- + t_2^+t_3^-\sigma^+\sigma^-D + t_1^+t_2^+(\upsilon^+\upsilon^-D + \upsilon^+\upsilon^-J) - \tfrac{1}{2}\left(t_1^+t_3^+ + (t_5^+)^2\right)\upsilon^0\upsilon^0J,\end{aligned} \tag{53}$$

subject to the conditions

$$\begin{aligned} t_2^+t_3^- &= t_2^-t_3^+ & t_1^\pm t_5^\pm &= 0 & t_4t_1^\pm &= 0\\ t_1^+t_2^+ &= t_1^-t_2^- & t_2^\pm t_5^\pm = t_2^\mp t_5^\pm &= 0 & t_4t_3^\pm &= 0\\ t_1^+t_3^+ - t_1^-t_3^- &= (t_5^- - t_5^+)(t_5^- + t_5^+) & t_3^\pm t_5^\pm &= 0 & t_4t_5^\pm &= 0. \end{aligned} \tag{54}$$

The case $t_4 = t_5^\pm = 0$ was already treated in Section 2.3 and proceeding in the same manner we arrive at the $d = 3$ version of Table 6. Hence those Lie algebras exist for all $d \geqslant 3$. It remains to classify graded conformal algebras which are unique to $d = 3$ and to this end we will assume from now on that at least one of t_4, t_5^+ and t_5^- is different from zero.

The last equation ($t_4t_5^\pm = 0$) in (54) implies that if $t_4 \neq 0$ then $t_5^\pm = 0$ and, viceversa, if at least one of $t_5^\pm$ is different from zero, then $t_4 = 0$. This breaks up the problem into several branches:

(1) $t_4 \neq 0$;
(2) $t_5^\pm \neq 0$;
(3) $t_5^+ \neq 0$ and $t_5^- = 0$; and
(4) $t_5^- \neq 0$ and $t_5^+ = 0$.

Since the last two branches are related by the automorphism (25), it is enough to consider one of them.

3.3.1. *The branch* $t_4 \neq 0$. In this case $t_1^\pm = t_3^\pm = t_5^\pm = 0$ and the deformation is

$$\mu = t_2^+\sigma^+\upsilon^-V_0 + t_2^-\sigma^-\upsilon^+V_0 + t_4\upsilon^+\upsilon^-V_0, \tag{55}$$

with Lie brackets

$$[S_+, V_{-a}] = t_2^+V_{0a}, \qquad [S_-, V_{+a}] = t_2^-V_{0a} \qquad \text{and} \qquad [V_{+a}, V_{-b}] = t_4\epsilon_{abc}V_{0c}, \tag{56}$$

for any $t_2^\pm$ and $t_4 \neq 0$. We can redefine V_0 and, without loss of generality, assume that $t_4 = 1$. If nonzero, the other two parameters $t_2^\pm$ can also be set to 1 by redefining $S_\pm$, respectively. The cases $(t_2^+, t_2^-) = (1, 0)$ and $(t_2^+, t_2^-) = (0, 1)$ are related by the automorphism (25), so they give isomorphic Lie algebras. In summary, we have three isomorphism classes of graded conformal algebras in this branch, which are listed in Table 10.

TABLE 10. Isomorphism classes of graded conformal algebras ($d = 3, t_4 \neq 0$)

t_2^+	t_2^-	Nonzero Lie brackets in addition to (3)		
0	0			$[V_+, V_-] = V_0$
1	0	$[S_+, V_-] = V_0$		$[V_+, V_-] = V_0$
1	1	$[S_+, V_-] = V_0$	$[S_-, V_+] = V_0$	$[V_+, V_-] = V_0$

3.3.2. *The branch* $t_4 = 0$ *and* $t_5^+t_5^- \neq 0$. In this case, $t_1^\pm = t_2^\pm = t_3^\pm = t_4 = 0$ and also $(t_5^+)^2 = (t_5^-)^2 \neq 0$. The deformation becomes

$$\mu = t_5^+\upsilon^0\upsilon^+V_+ + t_5^-\upsilon^0\upsilon^-V_- - \tfrac{1}{2}(t_5^+)^2\upsilon^0\upsilon^0J. \tag{57}$$

If $t_5^+ = t_5^-$, then we can rescale V_0 to arrive at

$$\mu = \upsilon^0\upsilon^+V_+ + \upsilon^0\upsilon^-V_- - \tfrac{1}{2}\upsilon^0\upsilon^0J, \tag{58}$$

whereas if $t_5^- = -t_5^+$, then under the same rescaling,

$$\mu = \upsilon^0\upsilon^+V_+ - \upsilon^0\upsilon^-V_- - \tfrac{1}{2}\upsilon^0\upsilon^0J. \tag{59}$$

In summary, we have two isomorphism classes of Lie algebras

$$[V_{0a}, V_{+b}] = \epsilon_{abc}V_{+c}, \qquad [V_{0a}, V_{-b}] = \varepsilon\epsilon_{abc}V_{-c} \qquad \text{and} \qquad [V_{0a}, V_{0b}] = -J_{ab}, \tag{60}$$

where $\varepsilon = \pm 1$.

3.3.3. *The branch* $t_4 = 0$ *and* $t_5^- = 0$. In this case $t_5^+ \neq 0$ and hence $t_1^+ = t_2^\pm = t_3^+ = 0$. The remaining integrability condition is

$$t_1^-t_3^- = (t_5^+)^2. \tag{61}$$

Therefore the deformation becomes

$$\mu = t_1^-\upsilon^0\upsilon^-S_- + t_3^-\sigma^-\upsilon^0V_- + t_5^+\upsilon^0\upsilon^+V_+ - \tfrac{1}{2}t_1^-t_3^-\upsilon^0\upsilon^0J, \tag{62}$$

where $t_1^- t_3^- = (t_5^+)^2 \neq 0$. The Lie brackets are

$$\begin{aligned}
[V_{0\,a}, V_{-\,b}] &= t_1^- \delta_{ab} S_- \\
[S_-, V_{0\,a}] &= t_3^- V_{-\,a} \\
[V_{0\,a}, V_{+\,b}] &= t_5^+ \epsilon_{abc} V_{+\,c} \\
[V_{0\,a}, V_{0\,b}] &= -t_1^- t_3^- J_{ab}.
\end{aligned} \tag{63}$$

We can rescale generators $V_0 \mapsto \frac{1}{t_5^+} V_0$, $S_- \mapsto \frac{t_1^-}{t_5^+} S_-$ and arrive at

$$\begin{aligned}
[V_{0\,a}, V_{-\,b}] &= \delta_{ab} S_- \\
[S_-, V_{0\,a}] &= V_{-\,a} \\
[V_{0\,a}, V_{+\,b}] &= \epsilon_{abc} V_{+\,c} \\
[V_{0\,a}, V_{0\,b}] &= -J_{ab}.
\end{aligned} \tag{64}$$

3.4. **Isomorphism classes of deformations.** We summarise the isomorphism classes of graded conformal algebras which are unique to $d = 3$: they all involve the ϵ_{abc} symbol in the Lie brackets. They are listed in Table 11, where the first column is the label used in this paper and which continues from Table 6.

TABLE 11. Isomorphism classes of graded conformal algebras unique to $d = 3$

Label	Nonzero Lie brackets in addition to (3)				
GCA_{16}				$[V_+, V_-] = V_0$	
GCA_{17}	$[S_+, V_-] = V_0$			$[V_+, V_-] = V_0$	
GCA_{18}	$[S_\pm, V_\mp] = V_0$			$[V_+, V_-] = V_0$	
GCA_{19}	$[S_-, V_0] = V_-$	$[V_0, V_+] = V_+$	$[V_0, V_-] = S_-$		$[V_0, V_0] = -J$
$\mathsf{GCA}_{20}^{(\varepsilon=\pm1)}$		$[V_0, V_+] = V_+$	$[V_0, V_-] = \varepsilon V_-$		$[V_0, V_0] = -J$

It is not hard to see that none of these Lie algebras admit an invariant inner product. Although now for $d = 3$ invariance under J and D allows for a component $\langle V_0, J\rangle$ in addition to those in equation (31), the argument in Section 2.7 still holds and non-degeneracy requires D appearing in $[S_+, S_-]$ and $[V_+, V_-]$, which is not the case for any of the Lie algebras in Table 11.

3.5. **Contractions.** No Lie algebra in Table 6 can contract to a Lie algebra in Table 11 because all Lie algebras in this latter table contain the Levi-Civita symbol ϵ_{abc} in at least one bracket, whereas no Lie algebra in the former table does. Any new contraction must therefore be from some Lie algebras in Table 11 to a Lie algebra in either table. All Lie algebras in Table 11 contract to the static graded conformal algebra GCA_1. One can work out the contractions relating the different algebras and display the result as a directed graph, as in Figure 1. Contractions can be composed, so the actual graph of contractions is the transitive closure of the graph in the figure. In Table 12 we give the values of the weights $a_\pm, b_\pm, c$ of $S_\pm$, $V_\pm$ and V_0 which are responsible for the different contractions, in the language of the discussion in Section 2.8. In the right-most part of the table, the contractions below the horizontal line involve graded conformal algebras unique to $d = 3$.

TABLE 12. Weights for contractions between graded conformal algebras

Contraction	a_+	a_-	b_+	b_-	c
$\mathsf{GCA}_{15} \to \mathsf{GCA}_{14}$	0	0	1	1	1
$\mathsf{GCA}_{15} \to \mathsf{GCA}_{13}$	1	0	1	0	0
$\mathsf{GCA}_{15} \to \mathsf{GCA}_{12}$	1	1	0	0	1
$\mathsf{GCA}_{15} \to \mathsf{GCA}_{8}$	1	1	2	0	1
$\mathsf{GCA}_{14} \to \mathsf{GCA}_{11}$	1	1	1	1	0
$\mathsf{GCA}_{14} \to \mathsf{GCA}_{10}$	1	1	0	0	1
$\mathsf{GCA}_{13} \to \mathsf{GCA}_{11}$	0	0	1	1	1
$\mathsf{GCA}_{13} \to \mathsf{GCA}_{9}$	1	1	0	0	1
$\mathsf{GCA}_{12} \to \mathsf{GCA}_{10}$	0	0	1	1	1
$\mathsf{GCA}_{12} \to \mathsf{GCA}_{9}$	1	1	1	1	0
$\mathsf{GCA}_{11} \to \mathsf{GCA}_{4}$	0	1	0	0	0

Contraction	a_+	a_-	b_+	b_-	c
$\mathsf{GCA}_{10} \to \mathsf{GCA}_{3}$	0	1	0	0	0
$\mathsf{GCA}_{9} \to \mathsf{GCA}_{2}$	0	0	1	0	0
$\mathsf{GCA}_{8} \to \mathsf{GCA}_{7}$	1	0	2	0	1
$\mathsf{GCA}_{8} \to \mathsf{GCA}_{6}$	0	1	0	1	0
$\mathsf{GCA}_{8} \to \mathsf{GCA}_{5}$	0	2	0	1	1
$\mathsf{GCA}_{7} \to \mathsf{GCA}_{4}$	0	0	0	1	0
$\mathsf{GCA}_{7} \to \mathsf{GCA}_{3}$	1	0	0	0	1
$\mathsf{GCA}_{6} \to \mathsf{GCA}_{4}$	0	0	1	0	1
$\mathsf{GCA}_{6} \to \mathsf{GCA}_{2}$	0	1	0	0	1
$\mathsf{GCA}_{5} \to \mathsf{GCA}_{3}$	1	0	0	0	1
$\mathsf{GCA}_{5} \to \mathsf{GCA}_{2}$	0	1	0	1	0

Contraction	a_+	a_-	b_+	b_-	c
$\mathsf{GCA}_{4} \to \mathsf{GCA}_{1}$	1	0	0	0	0
$\mathsf{GCA}_{3} \to \mathsf{GCA}_{1}$	1	0	0	0	0
$\mathsf{GCA}_{2} \to \mathsf{GCA}_{1}$	0	0	0	0	1
$\mathsf{GCA}_{20} \to \mathsf{GCA}_{1}$	0	0	0	0	1
$\mathsf{GCA}_{19} \to \mathsf{GCA}_{4}$	0	0	0	1	1
$\mathsf{GCA}_{19} \to \mathsf{GCA}_{2}$	0	1	0	0	1
$\mathsf{GCA}_{18} \to \mathsf{GCA}_{17}$	0	1	0	0	0
$\mathsf{GCA}_{18} \to \mathsf{GCA}_{10}$	0	0	1	1	1
$\mathsf{GCA}_{17} \to \mathsf{GCA}_{16}$	1	0	0	0	0
$\mathsf{GCA}_{17} \to \mathsf{GCA}_{3}$	0	0	1	0	0
$\mathsf{GCA}_{16} \to \mathsf{GCA}_{1}$	0	0	1	0	0

The simple conformal algebras $\mathfrak{so}(d+1, 2)$ and $\mathfrak{so}(d+2, 1)$ in dimension $d+1$ are isomorphic to the Lie algebra of isometries of the de Sitter spacetimes in dimension $d+2$. In other words, they are kinematical Lie algebras with $(d+1)$-dimensional space isotropy. The kinematical contractions of these algebras are contained among the kinematical Lie algebras in [2] (since $d+1 \geqslant 4$ for $d \geqslant 3$). It is therefore a natural question to ask whether any of the other Lie algebras in Table 6 can be interpreted as a kinematical Lie algebra in $d+1$, although perhaps with

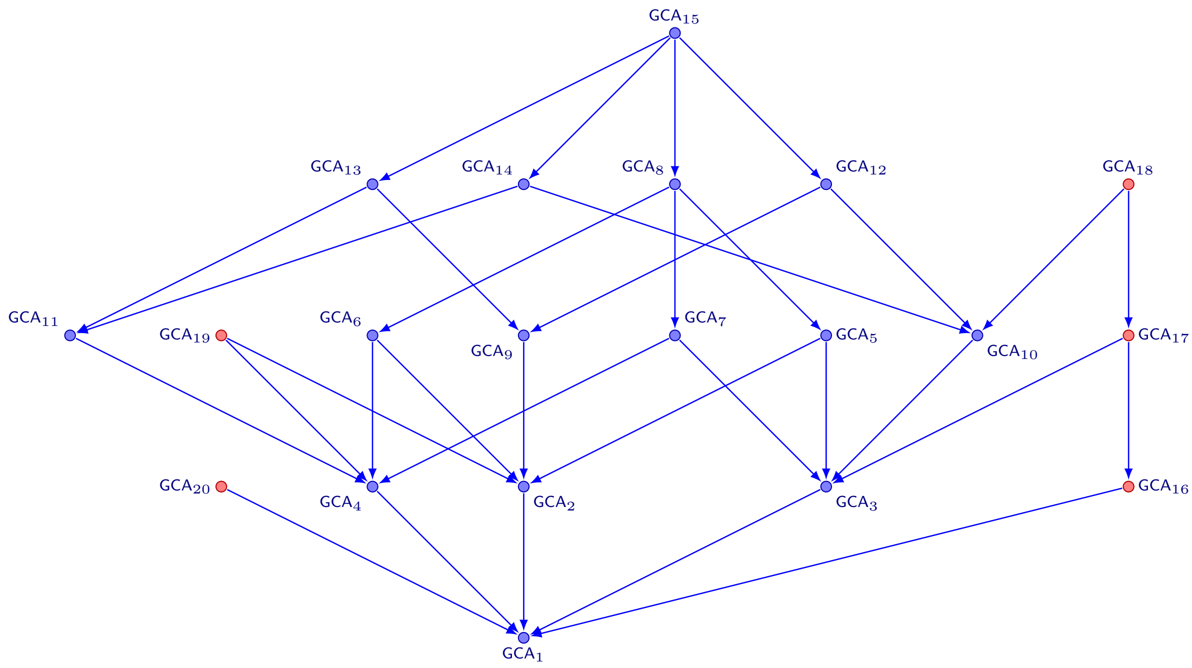

FIGURE 1. Contractions between graded conformal algebras: red dots are unique to $d = 3$.

$\mathfrak{so}(d, 1)$ isotropy as opposed to $\mathfrak{so}(d+1)$. Let $\mu = (a, \natural)$, where $a = 1, \dots, d$, and let $J_{\mu\nu}$, B_μ, P_μ and H generate a kinematical Lie algebra. This means that, in particular, the following brackets exist:

$$\begin{aligned}
[J_{\mu\nu}, J_{\rho\sigma}] &= \eta_{\nu\rho} J_{\mu\sigma} - \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\sigma} J_{\mu\rho} + \eta_{\mu\sigma} J_{\nu\rho}\\
[J_{\mu\nu}, B_\rho] &= \eta_{\nu\rho} B_\mu - \eta_{\mu\rho} B_\nu\\
[J_{\mu\nu}, P_\rho] &= \eta_{\nu\rho} P_\mu - \eta_{\mu\rho} P_\nu,
\end{aligned} \tag{65}$$

where $\eta_{ab} = \delta_{ab}$ and $\eta_{\natural\natural} = \pm 1$. Breaking the symmetry down to $\mathfrak{so}(d)$, we find that

(1) J_{ab} generate an $\mathfrak{so}(d)$ subalgebra,
(2) $J_{a\natural}$, B_a and P_a are vectors,
(3) $B_\natural$, $P_\natural$ and H are scalars,
(4) and we have the following additional Lie brackets:

$$\begin{aligned}
[J_{a\natural}, J_{b\natural}] &= -\eta_{\natural\natural} J_{ab}\\
[J_{a\natural}, B_b] &= -\delta_{ab} B_\natural\\
[J_{a\natural}, B_\natural] &= \eta_{\natural\natural} B_a\\
[J_{a\natural}, P_b] &= -\delta_{ab} P_\natural\\
[J_{a\natural}, P_\natural] &= \eta_{\natural\natural} P_a.
\end{aligned} \tag{66}$$

In summary, there is a vector V such that (ignoring signs,...) $[V, V] = J$ and additional vectors V', V'' and scalars S', S'' such that $[V, V'] = S'$, $[V, V''] = S''$, $[V, S'] = V'$ and $[V, S''] = V''$. Inspecting Tables 6 and 11, we see that apart from the de Sitter algebras, the only other Lie algebras satisfying this condition are $\mathsf{GCA}_{13}^{(\varepsilon)}$. To identify this Lie algebra, we let $J_{a\natural} := V_{0\,a}$, $B_a := V_{+\,a}$, $P_a := V_{-\,a}$, $B_\natural := -\varepsilon S^+$, $P_\natural := -\varepsilon S^-$ and $H := D$ and we see that this is kinematical with $\eta_{\natural\natural} = \varepsilon$. The only nonzero brackets in addition to the kinematical ones are

$$[H, B] = B \qquad \text{and} \qquad [H, P] = -P, \tag{67}$$

which means that it is isomorphic to a Newton-Hooke algebra (but in $d + 2$ dimensions) if $\varepsilon = 1$ and to a pseudo-Newton-Hooke algebra if $\varepsilon = -1$.

4. DEFORMATIONS FOR $d = 2$

As explained in the context of kinematical Lie algebras in [3], when $d = 2$ it is convenient to work with the complexified Lie algebra. The reason is two-fold: first of all, the vector representation of $\mathfrak{so}(2)$ has a larger endomorphism ring than that of $\mathfrak{so}(d)$ for any $d > 2$. This is because $\mathfrak{so}(2)$ is abelian and extends the endomorphism ring from $\mathbb{R}$ to $\mathbb{C}$. It is convenient to complexify so that $\mathbb{C}$ acts by complex multiplication. The second reason, which is the same reason in disguise, is that over the complex numbers we can diagonalise the action of J.

4.1. **The complex Lie algebra.** Let $\mathfrak{g}_{\mathbb{C}}$ be the complex Lie algebra spanned by $J, D, \mathbb{V}_0 := V_{01} + iV_{02}, \overline{\mathbb{V}}_0 := V_{01} - iV_{02}, \mathbb{V}_{\pm} := V_{\pm 1} + iV_{\pm 2}, \overline{\mathbb{V}}\pm := V_{\pm 1} - iV_{\pm 2}, S_{\pm}$, subject to the Lie brackets

$$\begin{aligned} [J, \mathbb{V}_{\pm}] &= -i\mathbb{V}_{\pm} \\ [J, \mathbb{V}_0] &= -i\mathbb{V}_0 \\ [J, \overline{\mathbb{V}}_{\pm}] &= i\overline{\mathbb{V}}_{\pm} \\ [J, \overline{\mathbb{V}}_0] &= i\overline{\mathbb{V}}_0 \end{aligned} \qquad \text{and} \qquad \begin{aligned} [D, \mathbb{V}_{\pm}] &= \pm\mathbb{V}_{\pm} \\ [D, \overline{\mathbb{V}}_{\pm}] &= \pm\overline{\mathbb{V}}_{\pm} \\ [D, S_{\pm}] &= \pm S_{\pm}. \end{aligned} \tag{68}$$

The complex Lie algebra $\mathfrak{g}_{\mathbb{C}}$ is the complexification of $\mathfrak{g}$ which is the real Lie algebra fixed under the antilinear involutive homomorphism $\star$ defined by

$$\star J = J, \quad \star D = D, \quad \star\mathbb{V}_0 = \overline{\mathbb{V}}_0, \quad \star\mathbb{V}_{\pm} = \overline{\mathbb{V}}_{\pm} \quad \text{and} \quad \star S_{\pm} = S_{\pm}. \tag{69}$$

4.2. **The deformation complex.** The deformation complex for $\mathfrak{g}_{\mathbb{C}}$ has cochains

$$C^p_{\mathbb{C}} := \operatorname{Hom}(\Lambda^p \mathbb{W}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}})^{\mathfrak{h}_{\mathbb{C}}}, \tag{70}$$

where $\mathbb{W}_{\mathbb{C}}$ is the complex span of $\mathbb{V}_0, \overline{\mathbb{V}}_0, \mathbb{V}_{\pm}, \overline{\mathbb{V}}_{\pm}, S_{\pm}$ and $\mathfrak{h}_{\mathbb{C}}$ is the complex abelian Lie algebra spanned by J, D. The differential $\partial : C^p_{\mathbb{C}} \to C^{p+1}_{\mathbb{C}}$ is defined on generators by

$$\begin{aligned} \partial J &= i\nu^+\mathbb{V}_+ - i\overline{\nu}^+\overline{\mathbb{V}}_+ + i\nu^-\mathbb{V}_- - i\overline{\nu}^-\overline{\mathbb{V}}_- + i\nu^0\mathbb{V}_0 - i\overline{\nu}^0\overline{\mathbb{V}}_0 \\ \partial D &= -\nu^+\mathbb{V}_+ - \overline{\nu}^+\overline{\mathbb{V}}_+ + \nu^-\mathbb{V}_- + \overline{\nu}^-\overline{\mathbb{V}}_- - \sigma^+S_+ + \sigma^-S_- \end{aligned} \tag{71}$$

and zero on other generators, where $\nu^0, \overline{\nu}^0, \nu^{\pm}, \overline{\nu}^{\pm}, \sigma^{\pm}$ are the canonical dual basis for $\mathbb{W}^*_{\mathbb{C}}$.

We shall now enumerate the first few spaces of cochains. To ensure that we do no leave any cochains out, we can use character theory to calculate the dimension of these spaces. The $\mathfrak{h}_{\mathbb{C}}$ character of $\mathbb{W}_{\mathbb{C}}$ is given by

$$\chi_{\mathbb{W}_{\mathbb{C}}}(q, \tau) = \underbrace{(\tau + \tau^{-1})}_{S_{\pm}} + \underbrace{(\tau + \tau^{-1})(q + q^{-1})}_{\mathbb{V}_{\pm}, \overline{\mathbb{V}}_{\pm}} + \underbrace{(q + q^{-1})}_{\mathbb{V}_0, \overline{\mathbb{V}}_0}, \tag{72}$$

and that of $\mathfrak{g}_{\mathbb{C}}$ is given by

$$\chi_{\mathfrak{g}_{\mathbb{C}}}(q, \tau) = 2 + (\tau + \tau^{-1}) + (\tau + \tau^{-1})(q + q^{-1}) + (q + q^{-1}), \tag{73}$$

where the 2 is due to J and D. The dimension of $C^p_{\mathbb{C}}$ is the constant term in

$$\chi_{\Lambda^p \mathbb{W}^*_{\mathbb{C}} \otimes \mathfrak{g}_{\mathbb{C}}}(q, \tau) = \chi_{\Lambda^p \mathbb{W}^*_{\mathbb{C}}}(q, \tau)\chi_{\mathfrak{g}_{\mathbb{C}}}(q, \tau). \tag{74}$$

To calculate this, we observe that $\chi_{\Lambda^p \mathbb{W}^*_{\mathbb{C}}}(q, \tau) = \chi_{\Lambda^p \mathbb{W}_{\mathbb{C}}}(q^{-1}, \tau^{-1}) = \chi_{\Lambda^p \mathbb{W}_{\mathbb{C}}}(q, \tau)$, where the last equality follows by inspection. The generating function for the characters of the exterior powers is given by

$$\sum_{p=0}^{\infty} t^p \chi_{\Lambda^p \mathbb{W}_{\mathbb{C}}}(q, \tau) = \exp\left(-\sum_{\ell=1}^{\infty} \frac{(-t)^{\ell}}{\ell} \chi_{\mathbb{W}_{\mathbb{C}}}(q^{\ell}, \tau^{\ell})\right). \tag{75}$$

Expanding and collecting terms, we find the following for the first few spaces of cochains:

$$\dim C^0_{\mathbb{C}} = 2, \qquad \dim C^1_{\mathbb{C}} = 8, \qquad \dim C^2_{\mathbb{C}} = 20 \qquad \text{and} \qquad \dim C^3_{\mathbb{C}} = 36, \tag{76}$$

which we proceed to enumerate:

$$\begin{aligned} C^0_{\mathbb{C}} &= \operatorname{span}_{\mathbb{C}}(J, D) \\ C^1_{\mathbb{C}} &= \operatorname{span}_{\mathbb{C}}(\nu^+\mathbb{V}_+, \overline{\nu}^+\overline{\mathbb{V}}_+, \nu^-\mathbb{V}_-, \overline{\nu}^-\overline{\mathbb{V}}_-, \nu^0\mathbb{V}_0, \overline{\nu}^0\overline{\mathbb{V}}_0, \sigma^+S_+, \sigma^-S_-) \\ C^2_{\mathbb{C}} &= \operatorname{span}_{\mathbb{C}}(\mathbb{c}_1, \overline{\mathbb{c}}_1, \mathbb{c}_2, \overline{\mathbb{c}}_2, \ldots, \mathbb{c}_8, \overline{\mathbb{c}}_8, \mathbb{c}_9, \mathbb{c}_{10}, \mathbb{c}_{11}, \mathbb{c}_{12}) \\ C^3_{\mathbb{C}} &= \operatorname{span}_{\mathbb{C}}(\mathbb{b}_1, \overline{\mathbb{b}}_1, \mathbb{b}_2, \overline{\mathbb{b}}_2, \ldots, \mathbb{b}_{16}, \overline{\mathbb{b}}_{16}, \mathbb{b}_{17}, \mathbb{b}_{18}, \mathbb{b}_{19}, \mathbb{b}_{20}), \end{aligned} \tag{77}$$

where (omitting $\otimes$ and $\wedge$),

$$\begin{aligned} &\mathbb{c}_1 = \nu^0\overline{\nu}^+S_+ && \mathbb{c}_4 = \sigma^-\nu^+\mathbb{V}_0 && \mathbb{c}_7 = \nu^+\overline{\nu}^-J && \mathbb{c}_{10} = \sigma^+\sigma^-D \\ &\mathbb{c}_2 = \nu^0\overline{\nu}^-S_- && \mathbb{c}_5 = \sigma^+\nu^0\mathbb{V}_+ && \mathbb{c}_8 = \nu^+\overline{\nu}^-D && \mathbb{c}_{11} = i\nu^0\overline{\nu}^0J \\ &\mathbb{c}_3 = \sigma^+\nu^-\mathbb{V}_0 && \mathbb{c}_6 = \sigma^-\nu^0\mathbb{V}_- && \mathbb{c}_9 = \sigma^+\sigma^-J && \mathbb{c}_{12} = i\nu^0\overline{\nu}^0D, \end{aligned} \tag{78}$$

and

$$\begin{aligned} &\mathbb{b}_1 = \nu^+\nu^-\overline{\nu}^+\mathbb{V}_+ && \mathbb{b}_6 = \nu^0\nu^-\overline{\nu}^+\mathbb{V}_0 && \mathbb{b}_{11} = \sigma^+\nu^+\overline{\nu}^-S_+ && \mathbb{b}_{16} = \sigma^-\nu^0\overline{\nu}^+D \\ &\mathbb{b}_2 = \nu^+\nu^-\overline{\nu}^-\mathbb{V}_- && \mathbb{b}_7 = \nu^0\nu^-\overline{\nu}^0\mathbb{V}_- && \mathbb{b}_{12} = \sigma^-\nu^-\overline{\nu}^+S_- && \mathbb{b}_{17} = i\sigma^+\nu^0\overline{\nu}^0S_+ \\ &\mathbb{b}_3 = \nu^+\nu^-\overline{\nu}^0\mathbb{V}_0 && \mathbb{b}_8 = \sigma^+\sigma^-\nu^+\mathbb{V}_+ && \mathbb{b}_{13} = \sigma^+\nu^0\overline{\nu}^-J && \mathbb{b}_{18} = i\sigma^-\nu^0\overline{\nu}^0S_- \\ &\mathbb{b}_4 = \nu^0\nu^+\overline{\nu}^-\mathbb{V}_0 && \mathbb{b}_9 = \sigma^+\sigma^-\nu^-\mathbb{V}_- && \mathbb{b}_{14} = \sigma^+\nu^0\overline{\nu}^-D && \mathbb{b}_{19} = i\sigma^+\nu^-\overline{\nu}^-S_- \\ &\mathbb{b}_5 = \nu^0\nu^+\overline{\nu}^0\mathbb{V}_+ && \mathbb{b}_{10} = \sigma^+\sigma^-\nu^0\mathbb{V}_0 && \mathbb{b}_{15} = \sigma^-\nu^0\overline{\nu}^+J && \mathbb{b}_{20} = i\sigma^-\nu^+\overline{\nu}^+S_+. \end{aligned} \tag{79}$$

Here $\overline{\mathbb{c}}_i = \mathbb{c}_i$ for $i = 9, 10, 11, 12$ and $\overline{\mathbb{b}}_i = \mathbb{b}_i$ for $i = 17, 18, 19, 20$. For the other cochains, complex conjugation is as expected: e.g., $\overline{\mathbb{c}}_1 = \overline{\nu}^0 \nu^+ S_+ = -\nu^+ \overline{\nu}^0 S_+$, et cetera.

4.3. **Infinitesimal deformations.** The subspace of real cochains is a subcomplex and we are interested in its cohomology in degree 2. We observe that $\partial : C^1_{\mathbb{C}} \to C^2_{\mathbb{C}}$ is the zero map, so that the cohomology $H^2_{\mathbb{C}}$ coincides with the space $Z^2_{\mathbb{C}}$ of cocycles. The map $\partial : C^2_{\mathbb{C}} \to C^3_{\mathbb{C}}$ is such that $\mathbb{c}_1, \overline{\mathbb{c}}_1, \dots, \mathbb{c}_6, \overline{\mathbb{c}}_6$ are cocycles, and

$$\begin{aligned} \partial\mathbb{c}_7 &= i(\overline{\mathbb{b}}_1 - \mathbb{b}_2 + \mathbb{b}_4 + \overline{\mathbb{b}}_6) & \partial\mathbb{c}_{10} &= -(\mathbb{b}_8 + \overline{\mathbb{b}}_8) + (\mathbb{b}_9 + \overline{\mathbb{b}}_9) \\ \partial\mathbb{c}_8 &= \overline{\mathbb{b}}_1 - \mathbb{b}_2 - \mathbb{b}_{11} - \overline{\mathbb{b}}_{12} & \partial\mathbb{c}_{11} &= (\mathbb{b}_5 + \overline{\mathbb{b}}_5) + (\mathbb{b}_7 + \overline{\mathbb{b}}_7) \\ \partial\mathbb{c}_9 &= i(\mathbb{b}_8 - \overline{\mathbb{b}}_8) + i(\mathbb{b}_9 - \overline{\mathbb{b}}_9) + i(\mathbb{b}_{10} - \overline{\mathbb{b}}_{10}) & \partial\mathbb{c}_{12} &= i(\mathbb{b}_5 - \overline{\mathbb{b}}_5) - i(\mathbb{b}_7 - \overline{\mathbb{b}}_7) - \mathbb{b}_{17} + \mathbb{b}_{18}, \end{aligned} \tag{80}$$

which spans the space $B^3_{\mathbb{C}}$ of 3-coboundaries. For future use, we notice that there are some cochains in $C^3_{\mathbb{C}}$ which do not appear as components of any coboundary in $B^3_{\mathbb{C}}$: $\mathbb{b}_3, \mathbb{b}_{13}, \mathbb{b}_{14}, \mathbb{b}_{15}, \mathbb{b}_{16}$ (and their complex conjugates) and also $\mathbb{b}_{19}, \mathbb{b}_{20}$.

The space of infinitesimal deformations is

$$H^2_{\mathbb{C}} \cong Z^2_{\mathbb{C}} = \operatorname{span}_{\mathbb{C}}(\mathbb{c}_1, \overline{\mathbb{c}}_1, \dots, \mathbb{c}_6, \overline{\mathbb{c}}_6) \tag{81}$$

and hence the most general (real) infinitesimal deformation is given by

$$\varphi^{(1)} = \sum_{i=1}^{6} t_i \mathbb{c}_i + \sum_{i=1}^{6} \overline{t}_i \overline{\mathbb{c}}_i =: \psi + \overline{\psi} \tag{82}$$

for some $t_i \in \mathbb{C}$.

4.4. **Obstructions.** The first obstruction to integrability of the infinitesimal deformation $\varphi^{(1)}$ is given by the cohomology class of $\varphi^{(1)} \bullet \varphi^{(1)}$ in $H^3_{\mathbb{C}}$, where $\bullet$ is the complex-linear extension of the Lie-admissible product introduced in Section 2.3. We tabulate this product on the space of cocycles in Table 13.

TABLE 13. Some components of the Nijenhuis–Richardson product $\bullet : C^2_{\mathbb{C}} \times C^2_{\mathbb{C}} \to C^3_{\mathbb{C}}$

$\bullet$	$\mathbb{c}_1$	$\overline{\mathbb{c}}_1$	$\mathbb{c}_2$	$\overline{\mathbb{c}}_2$	$\mathbb{c}_3$	$\overline{\mathbb{c}}_3$	$\mathbb{c}_4$	$\overline{\mathbb{c}}_4$	$\mathbb{c}_5$	$\overline{\mathbb{c}}_5$	$\mathbb{c}_6$	$\overline{\mathbb{c}}_6$
$\mathbb{c}_1$					$-\mathbb{b}_6$	$\overline{\mathbb{b}}_3$				$-\overline{\mathbb{b}}_5$		
$\overline{\mathbb{c}}_1$					$\mathbb{b}_3$	$-\overline{\mathbb{b}}_6$			$-\mathbb{b}_5$			
$\mathbb{c}_2$							$-\mathbb{b}_4$	$-\overline{\mathbb{b}}_3$				$-\overline{\mathbb{b}}_7$
$\overline{\mathbb{c}}_2$							$-\mathbb{b}_3$	$-\overline{\mathbb{b}}_4$			$-\mathbb{b}_7$	
$\mathbb{c}_3$	$-\overline{\mathbb{b}}_{11}$		$-i\mathbb{b}_{19}$								$\mathbb{b}_9$	
$\overline{\mathbb{c}}_3$		$-\mathbb{b}_{11}$		$i\mathbb{b}_{19}$								$\overline{\mathbb{b}}_9$
$\mathbb{c}_4$	$-i\mathbb{b}_{20}$		$\mathbb{b}_{12}$						$-\mathbb{b}_8$			
$\overline{\mathbb{c}}_4$		$i\mathbb{b}_{20}$		$\overline{\mathbb{b}}_{12}$						$-\overline{\mathbb{b}}_8$		
$\mathbb{c}_5$		$i\mathbb{b}_{17}$					$\mathbb{b}_{10}$					
$\overline{\mathbb{c}}_5$	$-i\mathbb{b}_{17}$							$\overline{\mathbb{b}}_{10}$				
$\mathbb{c}_6$				$i\mathbb{b}_{18}$	$-\mathbb{b}_{10}$							
$\overline{\mathbb{c}}_6$			$-i\mathbb{b}_{18}$			$-\overline{\mathbb{b}}_{10}$						

Calculating the product $\varphi^{(1)} \bullet \varphi^{(1)}$ we can use that

$$\varphi^{(1)} \bullet \varphi^{(1)} = (\psi + \overline{\psi}) \bullet (\psi + \psi\overline{)} = \psi \bullet \psi + \psi \bullet \overline{\psi} + \text{c.c.} \tag{83}$$

to arrive at

$$\begin{aligned} \varphi^{(1)} \bullet \varphi^{(1)} &= (t_3\overline{t}_1 - \overline{t}_2 t_4)\mathbb{b}_3 - t_2 t_4 \mathbb{b}_4 - \overline{t}_1 t_5 \mathbb{b}_5 - t_1 t_3 \mathbb{b}_6 - \overline{t}_2 t_6 \mathbb{b}_7 - t_4 t_5 \mathbb{b}_8 + t_3 t_6 \mathbb{b}_9 \\ &\quad + (t_4 t_5 - t_3 t_6)\mathbb{b}_{10} - \overline{t}_1 \overline{t}_3 \mathbb{b}_{11} + t_2 t_4 \mathbb{b}_{12} + i\overline{t}_1 t_5 \mathbb{b}_{17} + i\overline{t}_2 t_6 \mathbb{b}_{18} - i t_2 t_3 \mathbb{b}_{19} - i t_1 t_4 \mathbb{b}_{20} + \text{c.c.} \end{aligned} \tag{84}$$

We need to cancel this with a coboundary $\partial\varphi^{(2)}$, where

$$\varphi^{(2)} = u_1 \mathbb{c}_7 + \overline{u}_1 \overline{\mathbb{c}}_7 + u_2 \mathbb{c}_8 + \overline{u}_2 \overline{\mathbb{c}}_8 + u_3 \mathbb{c}_9 + u_4 \mathbb{c}_{10} + u_5 \mathbb{c}_{11} + u_6 \mathbb{c}_{12}, \tag{85}$$

with $u_1, u_2 \in \mathbb{C}$ and $u_3, \dots, u_6 \in \mathbb{R}$. Calculating $\partial\varphi^{(2)}$ we find

$$\begin{aligned} \partial\varphi^{(2)} &= (\overline{u}_2 - i\overline{u}_1)\mathbb{b}_1 - (u_2 + i u_1)\mathbb{b}_2 + i u_1 \mathbb{b}_4 + (u_5 + i u_6)\mathbb{b}_5 - i\overline{u}_1 \mathbb{b}_6 + (u_5 - i u_6)\mathbb{b}_7 - (u_4 - i u_3)\mathbb{b}_8 \\ &\quad + (u_4 + i u_3)\mathbb{b}_9 + i u_3 \mathbb{b}_{10} - u_2 \mathbb{b}_{11} + \overline{u}_2 \mathbb{b}_{12} - \tfrac{1}{2} u_6 \mathbb{b}_{17} + \tfrac{1}{2} u_6 \mathbb{b}_{18} + \text{c.c.} \end{aligned} \tag{86}$$

The first obstruction equation $\partial\varphi^{(2)} = \varphi^{(1)} \bullet \varphi^{(1)}$ implies

$$u_1 = it_2t_4, \quad u_2 = t_2t_4, \quad u_3 = 0, \quad u_4 = t_3t_6, \quad u_5 = -\bar{t}_1t_5 \quad \text{and} \quad u_6 = 0, \tag{87}$$

and results in the following relations between the t_i:

$$t_3t_6 = t_4t_5 \in \mathbb{R}, \quad \bar{t}_1t_5 = \bar{t}_2t_6 \in \mathbb{R}, \quad \bar{t}_1t_3 = \bar{t}_2t_4, \quad \bar{t}_1\bar{t}_3 = t_2t_4, \quad t_1t_4 \in \mathbb{R} \quad \text{and} \quad t_2t_3 \in \mathbb{R}. \tag{88}$$

The second obstruction equation is $\partial\varphi^{(3)} = \varphi^{(1)} \bullet \varphi^{(2)} + \varphi^{(2)} \bullet \varphi^{(1)}$, for which we need further components of the Nijenhuis–Richardson Lie-admissible product. Notice that $\varphi^{(2)} \bullet \varphi^{(1)}$ since there are no generators dual to J or D, so that we only need $\varphi^{(1)} \bullet \varphi^{(2)}$. The relevant products are tabulated in Table 14.

TABLE 14. Further components of the Nijenhuis–Richardson product $\bullet : C^2_{\mathbb{C}} \times C^2_{\mathbb{C}} \to C^3_{\mathbb{C}}$

$\bullet$	$\mathbb{c}_7$	$\bar{\mathbb{c}}_7$	$\mathbb{c}_8$	$\bar{\mathbb{c}}_8$	$\mathbb{c}_9$	$\mathbb{c}_{10}$	$\mathbb{c}_{11}$	$\mathbb{c}_{12}$
$\mathbb{c}_1$					$\mathbb{b}_{15}$	$\mathbb{b}_{16}$		
$\bar{\mathbb{c}}_1$					$\bar{\mathbb{b}}_{15}$	$\bar{\mathbb{b}}_{16}$		
$\mathbb{c}_2$					$-\mathbb{b}_{13}$	$-\mathbb{b}_{14}$		
$\bar{\mathbb{c}}_2$					$-\bar{\mathbb{b}}_{13}$	$-\bar{\mathbb{b}}_{14}$		
$\mathbb{c}_3$							$-i\bar{\mathbb{b}}_{13}$	$-i\bar{\mathbb{b}}_{14}$
$\bar{\mathbb{c}}_3$							$i\mathbb{b}_{13}$	$i\mathbb{b}_{14}$
$\mathbb{c}_4$							$-i\bar{\mathbb{b}}_{15}$	$-i\bar{\mathbb{b}}_{16}$
$\bar{\mathbb{c}}_4$							$i\mathbb{b}_{15}$	$i\mathbb{b}_{16}$
$\mathbb{c}_5$	$\mathbb{b}_{13}$		$\mathbb{b}_{14}$					
$\bar{\mathbb{c}}_5$		$\bar{\mathbb{b}}_{13}$		$\bar{\mathbb{b}}_{14}$				
$\mathbb{c}_6$		$-\mathbb{b}_{15}$		$-\mathbb{b}_{16}$				
$\bar{\mathbb{c}}_6$	$-\bar{\mathbb{b}}_{15}$		$-\bar{\mathbb{b}}_{16}$					

Using that $u_3 = u_6 = 0$, we find that

$$\varphi^{(1)} \bullet \varphi^{(2)} = (u_1t_5 + i\bar{t}_3u_5)\mathbb{b}_{13} + (t_5u_2 - t_2u_4)\mathbb{b}_{14} + (i\bar{t}_4u_5 - t_6\bar{u}_1)\mathbb{b}_{15} + (t_1u_4 - t_6\bar{u}_2)\mathbb{b}_{16} + \text{c.c.} \tag{89}$$

Using equation (88) we see that $\varphi^{(1)} \bullet \varphi^{(2)} = 0$ identically. Therefore we can take $\varphi^{(3)} = 0$. Since $\varphi^{(2)} \bullet \varphi^{(2)} = 0$, we also have $\varphi^{(4)} = 0$ and indeed all higher $\varphi^{(i)} = 0$ as well.

In summary, the most general deformation is

$$\varphi = \sum_{i=1}^{6} t_i\mathbb{c}_i + \sum_{i=1}^{6} \bar{t}_i\bar{\mathbb{c}}_i + it_2t_4\mathbb{c}_7 - i\bar{t}_2\bar{t}_4\bar{\mathbb{c}}_7 + t_2t_4\mathbb{c}_8 + \bar{t}_2\bar{t}_4\bar{\mathbb{c}}_8 + t_3t_6\mathbb{c}_{10} - \bar{t}_1t_5\mathbb{c}_{11} \tag{90}$$

subject to the integrability conditions in equation (88).

4.5. **Isomorphism classes of deformations.** The first three quadratic equations in (88) are equivalent to the vanishing of the following determinants:

$$\begin{vmatrix} t_4 & t_6 \\ t_3 & t_5 \end{vmatrix} \qquad \begin{vmatrix} \bar{t}_1 & t_6 \\ \bar{t}_2 & t_5 \end{vmatrix} \qquad \begin{vmatrix} \bar{t}_1 & t_4 \\ \bar{t}_2 & t_3 \end{vmatrix}, \tag{91}$$

which is equivalent to the columns being collinear. In other words, these equations imply that there exists $(x, y) \in \mathbb{C}^2 \setminus \{(0, 0)\}$ and $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$ such that

$$(\bar{t}_1, \bar{t}_2) = \alpha_1(x, y), \qquad (t_4, t_3) = \alpha_2(x, y) \qquad \text{and} \qquad (t_6, t_5) = \alpha_3(x, y). \tag{92}$$

The fourth quadratic equation in (88) says that

$$t_1t_3 = \bar{t}_2\bar{t}_4 \iff (\bar{\alpha}_1\alpha_2 - \alpha_1\bar{\alpha}_2)\bar{x}y = 0. \tag{93}$$

The reality conditions in (88) say that the following are real:

$$\alpha_2\alpha_3xy, \qquad \alpha_1\alpha_3xy, \qquad \bar{\alpha}_1\alpha_2|x|^2 \qquad \text{and} \qquad \bar{\alpha}_1\alpha_2|y|^2. \tag{94}$$

Since $(x, y) \neq (0, 0)$, $|x|^2 + |y|^2 \neq 0$, and therefore $\bar{\alpha}_1\alpha_2 \in \mathbb{R}$, which implies equation (93).

In summary, the integrability conditions are equivalent to equation (92) and the reality conditions

$$\bar{\alpha}_1\alpha_2 \in \mathbb{R}, \qquad \alpha_2\alpha_3xy \in \mathbb{R} \qquad \text{and} \qquad \alpha_1\alpha_3xy \in \mathbb{R}. \tag{95}$$

We will now analyse the different branches of solutions of these equations.

Since $(x, y) \neq (0, 0)$, we have three possibilities:

(1) $y = 0$ (and hence $x \neq 0$),
(2) $x = 0$ (and hence $y \neq 0$), and

(3) $xy \neq 0$.

The first two are related by the involutive automorphism τ of $\mathfrak{g}_{\mathbb{C}}$, defined by

$$\tau : (J, D, \mathbb{V}_0, \mathbb{V}_\pm, S_\pm) \mapsto (J, -D, \mathbb{V}_0, \mathbb{V}_\mp, S_\mp), \tag{96}$$

whose effect on the deformations is to exchange $x \leftrightarrow y$. Indeed, notice that on cochains τ exchanges

$$\mathbb{c}_1 \leftrightarrow \mathbb{c}_2, \qquad \mathbb{c}_3 \leftrightarrow \mathbb{c}_4 \qquad \text{and} \qquad \mathbb{c}_5 \leftrightarrow \mathbb{c}_6, \tag{97}$$

which, at the level of the parameters in the infinitesimal deformation, is equivalent to exchanging

$$t_1 \leftrightarrow t_2, \qquad t_3 \leftrightarrow t_4 \qquad \text{and} \qquad t_5 \leftrightarrow t_6, \tag{98}$$

which, from equation (92), can be seen to be equivalent to $x \leftrightarrow y$.

This leaves two branches, corresponding to (1) and (3).

4.5.1. *Branch* $y = 0$ *and* $x \neq 0$. Here $t_2 = t_3 = t_5 = 0$ and $t_1 = \overline{\alpha}_1 \overline{x}$, $t_4 = \alpha_2 x$ and $t_6 = \alpha_3 x$, with $\overline{\alpha}_1 \alpha_2 \in \mathbb{R}$. The corresponding deformation is

$$\varphi = \overline{\alpha}_1 \overline{x} \mathbb{c}_1 + \alpha_2 x \mathbb{c}_4 + \alpha_3 x \mathbb{c}_6 + \text{c.c.} \tag{99}$$

We still have the possibility of acting with automorphisms

$$\mathbb{V}_\pm \mapsto \lambda_\pm \mathbb{V}_\pm, \qquad \mathbb{V}_0 \mapsto \mu \mathbb{V}_0 \qquad \text{and} \qquad S_\pm \mapsto \xi_\pm S_\pm, \tag{100}$$

where $\lambda_\pm, \mu \in \mathbb{C}^\times$ and $\xi_\pm \in \mathbb{R}^\times$. The effect on cochains is

$$\mathbb{c}_1 \mapsto \frac{\xi_+}{\mu \overline{\lambda}_+} \mathbb{c}_1, \qquad \mathbb{c}_4 \mapsto \frac{\mu}{\xi_- \lambda_+} \mathbb{c}_4 \qquad \text{and} \qquad \mathbb{c}_6 \mapsto \frac{\lambda_-}{\xi_- \mu} \mathbb{c}_6, \tag{101}$$

and hence on parameters

$$t_1 \mapsto \frac{\mu \overline{\lambda}_+}{\xi_+} t_1, \qquad t_4 \mapsto \frac{\xi_- \lambda_+}{\mu} t_4 \qquad \text{and} \qquad t_6 \mapsto \frac{\xi_- \mu}{\lambda_-} t_6. \tag{102}$$

We now claim that if $\alpha_i x$ is different from zero, we may bring it to 1. Indeed, suppose that $\alpha_1 \neq 0$. then let $\mu = \frac{\xi_+}{\overline{\lambda}_+ \overline{\alpha}_1 \overline{x}}$. If $\alpha_1 = 0$, then μ remains unconstrained. Suppose that $\alpha_2 \neq 0$, then if $\alpha_1 = 0$ we choose $\mu = \xi_- \lambda_+ \alpha_2 x$, whereas if $\alpha_1 \neq 0$, then choose $\xi_- = \frac{\xi_+}{|\lambda_+|^2 |x|^2 \overline{\alpha}_1 \alpha_2}$, which is indeed real. Finally, if $\alpha_3 \neq 0$, then choose $\lambda_- = \xi_- \mu \alpha_3 x$.

In summary, we arrive at eight isomorphism classes of deformations in this branch summarised in Table 15.

TABLE 15. Isomorphism classes of deformations ($d = 2$, $y = 0$)

α_1	α_2	α_3	Deformation φ
0	0	0	0
1	0	0	$\nu^0 \overline{\nu}^+ S_+ - \nu^+ \overline{\nu}^0 S_+$
0	1	0	$\sigma^- \nu^+ \mathbb{V}_0 + \sigma^- \overline{\nu}^+ \overline{\mathbb{V}}_0$
0	0	1	$\sigma^- \nu^0 \mathbb{V}_- + \sigma^- \overline{\nu}^0 \overline{\mathbb{V}}_-$
1	1	0	$\nu^0 \overline{\nu}^+ S_+ + \sigma^- \nu^+ \mathbb{V}_0 - \nu^+ \overline{\nu}^0 S_+ + \sigma^- \overline{\nu}^+ \overline{\mathbb{V}}_0$
1	0	1	$\nu^0 \overline{\nu}^+ S_+ + \sigma^- \nu^0 \mathbb{V}_- - \nu^+ \overline{\nu}^0 S_+ + \sigma^- \overline{\nu}^0 \overline{\mathbb{V}}_-$
0	1	1	$\sigma^- \nu^+ \mathbb{V}_0 + \sigma^- \nu^0 \mathbb{V}_- + \sigma^- \overline{\nu}^+ \overline{\mathbb{V}}_0 + \sigma^- \overline{\nu}^0 \overline{\mathbb{V}}_-$
1	1	1	$\nu^0 \overline{\nu}^+ S_+ + \sigma^- \nu^+ \mathbb{V}_0 + \sigma^- \nu^0 \mathbb{V}_- - \nu^+ \overline{\nu}^0 S_+ + \sigma^- \overline{\nu}^+ \overline{\mathbb{V}}_0 + \sigma^- \overline{\nu}^0 \overline{\mathbb{V}}_-$

Working out the corresponding brackets, we find (up to the occasional rescaling and after applying the automorphism τ in (25)) the same list of Lie algebras as in Table 3. In other words, for this branch at least, the classification of $d = 2$ agrees with that of $d > 3$.

4.5.2. *Branch* $xy \neq 0$. In this branch we have

$$\begin{aligned} t_1 &= \overline{\alpha}_1 \overline{x} & & & t_2 &= \overline{\alpha}_1 \overline{y} & & & \overline{\alpha}_1 \alpha_2 &\in \mathbb{R} \\ t_3 &= \alpha_2 y & &\text{and} & t_4 &= \alpha_2 x & &\text{subject to} & \alpha_2 \alpha_3 x y &\in \mathbb{R} \\ t_5 &= \alpha_3 y & & & t_6 &= \alpha_3 x & & & \alpha_1 \alpha_3 x y &\in \mathbb{R}, \end{aligned} \tag{103}$$

and we have the possibility of applying the automorphisms in (100), whose effect on the t_i is as follows:

$$\begin{aligned} t_1 &\mapsto \frac{\mu \overline{\lambda}_+}{\xi_+} t_1 & & & t_2 &\mapsto \frac{\mu \overline{\lambda}_-}{\xi_-} t_2 \\ t_3 &\mapsto \frac{\xi_+ \lambda_-}{\mu} t_3 & &\text{and} & t_4 &\mapsto \frac{\xi_- \lambda_+}{\mu} t_4 \\ t_5 &\mapsto \frac{\xi_+ \mu}{\lambda_+} t_5 & & & t_6 &\mapsto \frac{\xi_- \mu}{\lambda_-} t_6. \end{aligned} \tag{104}$$

It is then straightforward to go through the eight possibilities, according to whether or not $\alpha_i = 0$ and bring these to normal forms which are tabulated in Table 16, where an asterisk in the α_i column means that it is not zero. As before, we have introduced $\varepsilon = \pm 1$, which arises whenever $\alpha_1\alpha_3 \neq 0$, and corresponds to the sign of the nonzero (real) number $\alpha_1\alpha_3 xy$.

TABLE 16. Isomorphism classes of deformations ($d = 2$, $xy \neq 0$)

α_1	α_2	α_3	Deformation φ
0	0	0	0
$\star$	0	0	$\mathbb{c}_1 + \mathbb{c}_2 + \text{c.c.}$
0	$\star$	0	$\mathbb{c}_3 + \mathbb{c}_4 + \text{c.c.}$
0	0	$\star$	$\mathbb{c}_5 + \mathbb{c}_6 + \text{c.c.}$
$\star$	$\star$	0	$\mathbb{c}_1 + \mathbb{c}_2 + \mathbb{c}_3 + \mathbb{c}_4 + i\mathbb{c}_7 + \mathbb{c}_8 + \text{c.c.}$
$\star$	0	$\star$	$(\mathbb{c}_1 + \mathbb{c}_2 + \varepsilon\mathbb{c}_5 + \varepsilon\mathbb{c}_6 + \text{c.c}) - \varepsilon\mathbb{c}_{11}$
0	$\star$	$\star$	$(\mathbb{c}_3 + \mathbb{c}_4 + \mathbb{c}_5 + \mathbb{c}_6 + \text{c.c}) + \mathbb{c}_{10}$
$\star$	$\star$	$\star$	$(\varepsilon\mathbb{c}_1 + \varepsilon\mathbb{c}_2 + \mathbb{c}_3 + \mathbb{c}_4 + \mathbb{c}_5 + \mathbb{c}_6 + i\varepsilon\mathbb{c}_7 + \varepsilon\mathbb{c}_8 + \text{c.c.}) + \mathbb{c}_{10} - \varepsilon\mathbb{c}_{11}$

Substituting the definitions (78) for the cochains and working out the Lie brackets, we find (up to the occasional rescaling) the same list of Lie algebras as in Table 5. In other words, the deformation problem for $d = 2$ has the same solution *mutatis mutandis* as in $d > 3$, resulting in the same list of isomorphism classes of Lie algebras in Table 6.

5. Central extensions

It is a natural question to ask whether a given Lie algebra admits central extensions, given the important rôle they play in applications. Central extensions of a Lie algebra[1] $\mathfrak{g}$ are classified by the second cohomology group $H^2(\mathfrak{g})$ with values in the trivial one-dimensional representation.

5.1. **Central extensions for $d \geqslant 3$.**

Let $d \geqslant 3$. All graded conformal algebras have a semisimple subalgebra $\mathfrak{h}$ isomorphic to $\mathfrak{so}(d)$: namely, the span of J_{ab}. The factorisation theorem of Hochschild and Serre [12] implies that $H^2(\mathfrak{g}) \cong H^2(\mathfrak{g}, \mathfrak{h})$, the relative cohomology group computed from the complex of $\mathfrak{h}$-invariant cochains with no legs along $\mathfrak{h}$. Since $\mathfrak{h}$ lies in degree 0 in every graded conformal algebra $\mathfrak{g}$, the complex $C^\bullet(\mathfrak{g}, \mathfrak{h})$ breaks up into the direct sum of subcomplexes of a fixed degree. Since the grading element D acts trivially on cohomology and acts reducibly in the complex, we have further that $H^2(\mathfrak{g}, \mathfrak{h}) \cong H^{0,2}(\mathfrak{g}, \mathfrak{h})$, which can be computed from the degree-0 piece of the complex. This complex is

$$C^{0,p}(\mathfrak{g}, \mathfrak{h}) \cong (\Lambda^p(\mathfrak{g}/\mathfrak{h})^*)^{\deg 0}, \tag{105}$$

with the differential induced from that of the Chevalley–Eilenberg complex of $\mathfrak{g}$.

For the graded conformal algebras under discussion and letting δ denote the dual to D, we have that

$$C^{0,1}(\mathfrak{g}, \mathfrak{h}) = \operatorname{span}_{\mathbb{R}}(\delta) \qquad \text{and} \qquad C^{0,2}(\mathfrak{g}, \mathfrak{h}) = \operatorname{span}_{\mathbb{R}}(\sigma^+\sigma^- := \sigma^+ \wedge \sigma^-, \upsilon^+\upsilon^- := \upsilon^+_a \wedge \upsilon^-_a). \tag{106}$$

It is then a simple matter to go down Table 6 and calculate $\partial\delta$, $\partial(\sigma^+\sigma^-)$ and $\partial(\upsilon^+\upsilon^-)$ to determine $H^2(\mathfrak{g})$ and hence the possible central extensions.

5.1.1. GCA_1, GCA_3, GCA_{10}, GCA_{16}, GCA_{17} *and* GCA_{18}.

These share the same differential:

$$\partial\delta = 0, \qquad \partial\sigma^\pm = \mp\delta\sigma^\pm \qquad \text{and} \qquad \partial\upsilon^\pm = \mp\delta\upsilon^\pm, \tag{107}$$

which implies

$$\partial(\sigma^+\sigma^-) = 0 \qquad \text{and} \qquad \partial(\upsilon^+\upsilon^-) = 0, \tag{108}$$

and hence

$$H^2 \cong \operatorname{span}_{\mathbb{R}}(\sigma^+\sigma^-, \upsilon^+\upsilon^-). \tag{109}$$

This results in the additional brackets

$$[S_+, S_-] = Z_1 \qquad \text{and} \qquad [V_+, V_-] = Z_2, \tag{110}$$

where we have introduced two central generators Z_1 and Z_2.

[1]In this section, and for psychological reasons, $\mathfrak{g}$ shall denote a general Lie algebra, not necessarily the static graded conformal Lie algebra as it did in the sections where we discussed deformations.

5.1.2. GCA_2 *and* GCA_5. These share the same differential:

$$\partial\delta = 0, \qquad \partial\sigma^+ = -\delta\sigma^+, \qquad \partial\sigma^- = \delta\sigma^- - \upsilon^0\upsilon^- \qquad \text{and} \qquad \partial\upsilon^\pm = \mp\delta\upsilon^\pm, \tag{111}$$

so that

$$\partial(\sigma^+\sigma^+) = \sigma^+\upsilon^0\upsilon^- \qquad \text{and} \qquad \partial(\upsilon^+\upsilon^-) = 0, \tag{112}$$

and hence

$$H^2 \cong \operatorname{span}_{\mathbb{R}}(\upsilon^+\upsilon^-), \tag{113}$$

with additional Lie brackets

$$[\mathsf{V}_+, \mathsf{V}_-] = \mathsf{Z}. \tag{114}$$

5.1.3. GCA_4 *and* GCA_7. These share the same differential

$$\partial\delta = 0, \qquad \partial\sigma^\pm = \mp\delta\sigma^\pm, \qquad \partial\upsilon^- = \delta\upsilon^- \qquad \text{and} \qquad \partial\upsilon^+ = -\delta\upsilon^+ - \sigma^+\upsilon^0, \tag{115}$$

so that

$$\partial(\sigma^+\sigma^-) = 0 \qquad \text{and} \qquad \partial(\upsilon^+\upsilon^-) = -\sigma^+\upsilon^0\upsilon^-, \tag{116}$$

so that

$$H^2 \cong \operatorname{span}_{\mathbb{R}}(\sigma^+\sigma^-), \tag{117}$$

with additional brackets

$$[\mathsf{S}_+, \mathsf{S}_-] = \mathsf{Z}. \tag{118}$$

5.1.4. GCA_6 *and* GCA_8. These share the same differential

$$\partial\delta = 0, \qquad \partial\sigma^+ = -\delta\sigma^+, \qquad \partial\sigma^- = \delta\sigma^- - \upsilon^0\upsilon^-, \qquad \partial\upsilon^+ = -\delta\upsilon^+ - \sigma^=\upsilon^0 \qquad \text{and} \qquad \partial\upsilon^- = \delta\upsilon^-, \tag{119}$$

so that

$$\partial(\sigma^+\sigma^-) = \sigma^+\upsilon^0\upsilon^- \qquad \text{and} \qquad \partial(\upsilon^+\upsilon^-) = -\sigma^+\upsilon^0\upsilon^-, \tag{120}$$

so that

$$H^2 \cong \operatorname{span}_{\mathbb{R}}(\sigma^+\sigma^- + \upsilon^+\upsilon^-), \tag{121}$$

with additional brackets

$$[\mathsf{S}_+, \mathsf{S}_-] = \mathsf{Z} \qquad \text{and} \qquad [\mathsf{V}_+, \mathsf{V}_-] = \mathsf{Z}. \tag{122}$$

5.1.5. GCA_9. Here the differential is given by

$$\partial\delta = 0, \qquad \partial\sigma^\pm = \mp\delta\sigma^\pm - \upsilon^0\upsilon^\pm \qquad \text{and} \qquad \partial\upsilon^\pm = \mp\delta\upsilon^\pm, \tag{123}$$

so that

$$\partial(\sigma^+\sigma^-) = -\sigma^-\upsilon^0\upsilon^+ + \sigma^+\upsilon^0\upsilon^- \qquad \text{and} \qquad \partial(\upsilon^+\upsilon^-) = 0, \tag{124}$$

so that

$$H^2 \cong \operatorname{span}_{\mathbb{R}}(\upsilon^+\upsilon^-), \tag{125}$$

with additional brackets

$$[\mathsf{V}_+, \mathsf{V}_-] = \mathsf{Z}. \tag{126}$$

5.1.6. GCA_{11}. Here the differential is given by

$$\partial\delta = 0, \qquad \partial\sigma^\pm = \mp\delta\sigma^\pm \qquad \text{and} \qquad \partial\upsilon^\pm = \mp\delta\upsilon^\pm - \sigma^\pm\upsilon^0, \tag{127}$$

so that

$$\partial(\sigma^+\sigma^-) = 0 \qquad \text{and} \qquad \partial(\upsilon^+\upsilon^-) = -\sigma^+\upsilon^0\upsilon^- + \sigma^-\upsilon^0\upsilon^+, \tag{128}$$

so that

$$H^2 \cong \operatorname{span}_{\mathbb{R}}(\sigma^+\sigma^-), \tag{129}$$

with additional brackets

$$[\mathsf{S}_+, \mathsf{S}_-] = \mathsf{Z}. \tag{130}$$

5.1.7. GCA_{12}. Here the differential is given by

$$\partial\delta = -\upsilon^+\upsilon^-, \qquad \partial\sigma^\pm = \mp\delta\sigma^\pm - \upsilon^0\upsilon^\pm \qquad \text{and} \qquad \partial\upsilon^\pm = \mp\delta\upsilon^\pm, \tag{131}$$

so that

$$\partial(\sigma^+\sigma^-) = -\sigma^-\upsilon^0\upsilon^+ + \sigma^+\upsilon^0\upsilon^- \qquad \text{and} \qquad \partial(\upsilon^+\upsilon^-) = 0, \tag{132}$$

so that the only 2-cocycle is $\upsilon^+\upsilon^-$, which is also a coboundary, so that $H^2 = 0$ and GCA_{12} admits no (nontrivial) central extensions.

5.1.8. $\mathsf{GCA}_{13}^{(\varepsilon)}$. The differential is given by

$$\partial\delta = 0, \qquad \partial\sigma^\pm = \mp\delta\sigma^+ - \varepsilon\upsilon^0\upsilon^\pm \qquad \text{and} \qquad \partial\upsilon^\pm = \mp\delta\upsilon^\pm - \sigma^\pm\upsilon^0, \tag{133}$$

so that

$$\partial(\sigma^+\sigma^-) = \varepsilon\sigma^+\upsilon^0\upsilon^- - \varepsilon\sigma^-\upsilon^0\upsilon^+ \qquad \text{and} \qquad \partial(\upsilon^+\upsilon^-) = -\sigma^+\upsilon^0\upsilon^- + \sigma^-\upsilon^0\upsilon^+, \tag{134}$$

and hence

$$H^2 \cong \operatorname{span}_{\mathbb{R}}(\sigma^+\sigma^- + \varepsilon\upsilon^+\upsilon^-), \tag{135}$$

with additional brackets

$$[S_+, S_-] = Z \qquad \text{and} \qquad [V_+, V_-] = \varepsilon Z. \tag{136}$$

This central extension is isomorphic to the “pseudo-Bargmann” Lie algebra which has recently appeared in [13], where it is exhibited as a contraction of a trivial central extension of the de Sitter algebras in $d+2$ dimensions.

5.1.9. GCA_{14}. Here the differential is given by

$$\partial\delta = -\sigma^+\sigma^-, \qquad \partial\sigma^\pm = \mp\delta\sigma^\pm \qquad \text{and} \qquad \partial\upsilon^\pm = \mp\delta\upsilon^\pm - \sigma^\pm\upsilon^0, \tag{137}$$

so that

$$\partial(\sigma^+\sigma^-) = 0 \qquad \text{and} \qquad \partial(\upsilon^+\upsilon^-) = -\sigma^+\upsilon^0\upsilon^- + \sigma^-\upsilon^0\upsilon^+, \tag{138}$$

so that the only 2-cocycle is $\sigma^+\sigma^-$, which is also a coboundary, so that $H^2 = 0$ and GCA_{14} admits no (nontrivial) central extensions.

Lie algebra GCA_{14} is isomorphic to the galilean conformal algebra of [14], which was known not to admit a central extension. The construction in [14] is precisely the contraction described above from the de Sitter algebras.

5.1.10. $\mathsf{GCA}_{15}^{(\varepsilon)}$. These are semisimple Lie algebras and, by the second Whitehead Lemma, the second cohomology $H^2 = 0$.

5.1.11. GCA_{19}. Here the differential is given by

$$\begin{aligned} \partial\delta &= 0 \\ \partial\sigma^+ &= -\delta\sigma^+ \\ \partial\sigma^- &= \delta\sigma^- - \upsilon^0\upsilon^- \\ \partial\upsilon_a^+ &= -\delta\upsilon_a^+ - \epsilon_{abc}\upsilon_b^0\upsilon_c^+ \\ \partial\upsilon^- &= \delta\upsilon^- - \sigma^-\upsilon^0, \end{aligned} \tag{139}$$

so that

$$\partial(\sigma^+\sigma^-) = \sigma^+\upsilon^0\upsilon^- \qquad \text{and} \qquad \partial(\upsilon^+\upsilon^-) = \sigma^-\upsilon^0\upsilon^+ - \upsilon^0\upsilon^+\upsilon^-, \tag{140}$$

so that $H^2 = 0$ and this conformal algebra admits no (nontrivial) central extensions. This result also follows with little or no calculation by using Hochschild–Serre relative to the semisimple subalgebra spanned by J and V_0 which is isomorphic to $\mathfrak{so}(3) \oplus \mathfrak{so}(3)$. In this case, the relative complex has no cochains in degree 0.

5.1.12. $\mathsf{GCA}_{20}^{(\varepsilon)}$. The differential is given by

$$\begin{aligned} \partial\delta &= 0 \\ \partial\sigma^\pm &= \mp\delta\sigma^+ \\ \partial\upsilon_a^+ &= -\delta\upsilon_a^+ - \epsilon_{abc}\upsilon_b^0\upsilon_c^+ \\ \partial\upsilon_a^- &= \delta\upsilon_a^- = \delta\upsilon_a^- - \varepsilon\epsilon_{abc}\upsilon_b^0\upsilon_c^-, \end{aligned} \tag{141}$$

so that

$$\partial(\sigma^+\sigma^-) = 0 \qquad \text{and} \qquad \partial(\upsilon^+\upsilon^-) = (\varepsilon - 1)\upsilon^0\upsilon^+\upsilon^- \tag{142}$$

and hence

$$H^2 \cong \begin{cases} \operatorname{span}_{\mathbb{R}}(\sigma^+\sigma^-, \upsilon^+\upsilon^-) & \varepsilon = 1 \\ \operatorname{span}_{\mathbb{R}}(\sigma^+\sigma^-) & \varepsilon = -1, \end{cases} \tag{143}$$

with additional brackets

$$\begin{cases} [S_+, S_-] = Z_1 \quad \text{and} \quad [V_+, V_-] = Z_2 & \varepsilon = 1 \\ [S_+, S_-] = Z & \varepsilon = -1. \end{cases} \tag{144}$$

The results are summarised in Table 17.

TABLE 17. Central extensions of graded conformal algebras ($d \geqslant 3$)

Label	Additional central brackets		$\dim H^2$
GCA_1	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
GCA_2	$[V_+, V_-] = Z$		1
GCA_3	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
GCA_4		$[S_+, S_-] = Z$	1
GCA_5	$[V_+, V_-] = Z$		1
GCA_6	$[V_+, V_-] = Z$	$[S_+, S_-] = Z$	1
GCA_7		$[S_+, S_-] = Z$	1
GCA_8	$[V_+, V_-] = Z$	$[S_+, S_-] = Z$	1
GCA_9	$[V_+, V_-] = Z$		1
GCA_{10}	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
GCA_{11}		$[S_+, S_-] = Z$	1
GCA_{12}			0
$\mathsf{GCA}_{13}^{(\varepsilon=\pm1)}$	$[V_+, V_-] = \varepsilon Z$	$[S_+, S_-] = Z$	1
GCA_{14}			0
$\mathsf{GCA}_{15}^{(\varepsilon=\pm1)}$			0
GCA_{16}	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
GCA_{17}	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
GCA_{18}	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
GCA_{19}			0
$\mathsf{GCA}_{20}^{(\varepsilon=+1)}$	$[V_+, V_-] = Z_1$	$[S_+, S_-] = Z_2$	2
$\mathsf{GCA}_{20}^{(\varepsilon=-1)}$		$[S_+, S_-] = Z$	1

5.2. **Invariant inner products.** It often happens that a Lie algebra does not admit an invariant inner product, yet a central extension of it does. For this to happen, the degeneracy of the invariant inner product in the original Lie algebra should be curable by adding central elements. For the kind of Lie algebras under consideration, this happens if and only if the original Lie algebra has an invariant symmetric bilinear form with a one-dimensional kernel spanned by D. It will turn out that there is only one additional metric Lie algebra: a one-dimensional central extension of GCA_{16}, which only exists for $d = 3$. In this section we provide the details.

First of all we notice that (suppressing indices)

$$\langle V_+, V_- \rangle = \langle V_+, [V_-, J] \rangle = \langle [V_+, V_-], J \rangle , \tag{145}$$

so that unless $\langle [V_+, V_-], J \rangle \neq 0$, any invariant bilinear form is degenerate. Clearly no central term in $[V_+, V_-]$ contributes to $\langle [V_+, V_-], J \rangle$, so the only way $\langle [V_+, V_-], J \rangle$ can be nonzero is if $[V_+, V_-] = J + \cdots$ or, if $d = 3$, $[V_+, V_-] = V_0 + \cdots$. This only occurs for GCA_{12}, $\mathsf{GCA}_{15}^{(\varepsilon)}$, GCA_{16}, GCA_{17} and GCA_{18}. The Lie algebras GCA_{12} and $\mathsf{GCA}_{15}^{(\varepsilon)}$ do not admit central extensions, so we concentrate on the rest.

No central extension of the Lie algebras GCA_{17} and GCA_{18} admit an invariant inner product. Indeed, omitting indices,

$$\langle V_0, V_0 \rangle = \langle V_0, [V_0, J] \rangle = \langle [V_0, V_0], J \rangle = 0, \tag{146}$$

and

$$\langle J, V_0 \rangle = \langle J, [S_+, V_-] \rangle = \langle [J, S_+], V_- \rangle = 0. \tag{147}$$

Therefore $\langle V_0, - \rangle = 0$ and any invariant symmetric bilinear form is degenerate.

Finally, let us consider central extensions of GCA_{16}, whose Lie brackets, in addition to (3), are (omitting indices)

$$[V_+, V_-] = V_0 + Z_1 \qquad \text{and} \qquad [S_+, S_-] = Z_2. \tag{148}$$

The two-dimensional central extension does not admit an invariant inner product, but imposing a linear relation between Z_1 and Z_2 results in a metric Lie algebra. We may describe it as a double extension [15, 16] of the abelian Lie algebra $\mathfrak{a}$ with generators V_+, V_-, S_+, S_- with (trivially invariant) inner product

$$\langle V_{+a}, V_{-b} \rangle = \alpha \delta_{ab} \qquad \text{and} \qquad \langle S_+, S_- \rangle = \beta, \tag{149}$$

for $\alpha, \beta \neq 0$. The Lie algebra $\mathfrak{g} \cong \mathfrak{co}(3)$ spanned by $R_a = -\frac{1}{2}\epsilon_{abc}J_{bc}$ and D acts on $\mathfrak{a}$ via skew-symmetric derivations:

$$[R_a, V_{\pm b}] = \epsilon_{abc} V_{\pm c}, \qquad [D, V_{\pm a}] = \pm V_{\pm a} \qquad \text{and} \qquad [D, S_\pm] = \pm S_\pm. \tag{150}$$

To construct the double extension, we add generators $V_{0\,a}$ dual to R_a and Z dual to D, together with the dual pairings

$$\langle R_a, V_{0\,b}\rangle = \delta_{ab} \qquad \text{and} \qquad \langle D, Z\rangle = 1, \tag{151}$$

and extend the brackets of $\mathfrak{a}$ by

$$[V_{+\,a}, V_{-\,b}] = \alpha\epsilon_{abc}V_{0\,c} + \alpha\delta_{ab}Z \qquad \text{and} \qquad [S_+, S_-] = \beta Z. \tag{152}$$

Choosing $\alpha = 1$, we see that this is a quotient of the universal central extension of GCA_{16} where $Z_1 = Z$ and $Z_2 = \beta Z$. The most general invariant inner product is given by

$$\begin{aligned} \langle V_{+\,a}, V_{-\,b}\rangle &= \delta_{ab} & & & \langle D, Z\rangle &= 1 \\ \langle S_+, S_-\rangle &= \beta & &\text{and} & \langle D, D\rangle &= \lambda \\ \langle R_a, V_{0\,b}\rangle &= \delta_{ab} & & & \langle R_a, R_b\rangle &= \mu\delta_{ab}, \end{aligned} \tag{153}$$

for all $\lambda, \mu \in \mathbb{R}$ and $\beta \in \mathbb{R}^\times$.

5.3. **Central extensions for $d = 2$.** In $d = 2$ the calculation differs because $\mathfrak{so}(2)$ is not semisimple and the factorisation theorem does not apply. Nevertheless the two-dimensional abelian subalgebra $\mathfrak{r} = \operatorname{span}_{\mathbb{R}}(J, D)$ acts reducibly and trivially on cohomology, and therefore we may work with the subcomplex of $\mathfrak{r}$-invariant cochains. The 1- and 2-cochains are as follows:

$$\begin{aligned} C^1 &= \operatorname{span}_{\mathbb{R}}(\rho, \delta) \\ C^2 &= \operatorname{span}_{\mathbb{R}}(\rho\delta, \delta_{ab}\upsilon^+_a\upsilon^-_b, \epsilon_{ab}\upsilon^+_a\upsilon^-_b, \epsilon_{ab}\upsilon^0_a\upsilon^0_b, \sigma^+\sigma^-). \end{aligned} \tag{154}$$

The calculations are routine and we only list the result, which is summarised in Table 18.

TABLE 18. Central extensions of graded conformal algebras ($d = 2$)

Label	Additional central brackets				$\dim H^2$
GCA_1	$[J, D] = Z_1$	$[V_{+\,a}, V_{-\,b}] = \delta_{ab}Z_2 + \epsilon_{ab}Z_3$	$[V_{0\,a}, V_{0\,b}] = \epsilon_{ab}Z_4$	$[S_+, S_-] = Z_5$	5
GCA_2	$[J, D] = Z_1$	$[V_{+\,a}, V_{-\,b}] = \delta_{ab}Z_2 + \epsilon_{ab}Z_3$	$[V_{0\,a}, V_{0\,b}] = \epsilon_{ab}Z_4$		4
GCA_3	$[J, D] = Z_1$	$[V_{+\,a}, V_{-\,b}] = \delta_{ab}Z_2 + \epsilon_{ab}Z_3$		$[S_+, S_-] = Z_4$	4
GCA_4	$[J, D] = Z_1$		$[V_{0\,a}, V_{0\,b}] = \epsilon_{ab}Z_2$	$[S_+, S_-] = Z_3$	3
GCA_5	$[J, D] = Z_1$	$[V_{+\,a}, V_{-\,b}] = \delta_{ab}Z_2 + \epsilon_{ab}Z_3$			3
GCA_6	$[J, D] = Z_1$	$[V_{+\,a}, V_{-\,b}] = \delta_{ab}Z_2$	$[V_{0\,a}, V_{0\,b}] = \epsilon_{ab}Z_3$	$[S_+, S_-] = -Z_2$	3
GCA_7	$[J, D] = Z_1$	$[V_{+\,a}, V_{-\,b}] = \epsilon_{ab}Z_2$	$[V_{0\,a}, V_{0\,b}] = -\epsilon_{ab}Z_2$	$[S_+, S_-] = Z_3$	3
GCA_8	$[J, D] = Z_1$	$[V_{+\,a}, V_{-\,b}] = \delta_{ab}Z_2 + \epsilon_{ab}Z_3$	$[V_{0\,a}, V_{0\,b}] = -\epsilon_{ab}Z_3$	$[S_+, S_-] = Z_2$	3
GCA_9	$[J, D] = Z_1$	$[V_{+\,a}, V_{-\,b}] = \delta_{ab}Z_2 + \epsilon_{ab}Z_3$	$[V_{0\,a}, V_{0\,b}] = \epsilon_{ab}Z_4$		4
GCA_{10}	$[J, D] = Z_1$	$[V_{+\,a}, V_{-\,b}] = \delta_{ab}Z_2 + \epsilon_{ab}Z_3$		$[S_+, S_-] = Z_4$	4
GCA_{11}	$[J, D] = Z_1$		$[V_{0\,a}, V_{0\,b}] = \epsilon_{ab}Z_2$	$[S_+, S_-] = Z_3$	3
GCA_{12}					0
$\mathsf{GCA}_{13}^{(\varepsilon=\pm1)}$					0
GCA_{14}		$[V_{+\,a}, V_{-\,b}] = \epsilon_{ab}Z_1$	$[V_{0\,a}, V_{0\,b}] = -\epsilon_{ab}Z_1$	$[S_+, S_-] = Z_2$	2
$\mathsf{GCA}_{15}^{(\varepsilon=\pm1)}$					0

5.4. **Invariant inner products.** A natural question is whether there are any centrally extended Lie algebras for $d = 2$ which admit an invariant inner product. Again, there are some Lie algebras which do not admit central extensions: GCA_{12}, $\mathsf{GCA}_{13}^{(\varepsilon)}$ and $\mathsf{GCA}_{15}^{(\varepsilon)}$, and hence we will not consider them further. In $d = 2$ an invariant inner product must have $\langle V_0, V_0\rangle$ and $\langle V_+, V_-\rangle$ nonzero, but (omitting indices)

$$\langle V_+, V_-\rangle = \langle V_+, [V_-, J]\rangle = \langle [V_+, V_-], J\rangle \tag{155}$$

and, similarly,

$$\langle V_0, V_0\rangle = \langle V_0, [V_0, J]\rangle = \langle [V_0, V_0], J\rangle\,, \tag{156}$$

where now any of J, D, Z_i can have nonzero inner product with J.

Every central extension of GCA_4 and GCA_{11} has $[V_+, V_-] = 0$ and every central extension of GCA_{10} has $[V_0, V_0] = 0$, hence they cannot be metric. Neither can any central extension of GCA_7 or GCA_{14} because (omitting indices)

$$\langle V_0, V_0\rangle = \langle V_0, [S_+, V_-]\rangle = \langle [V_-, V_0], S_+\rangle = 0. \tag{157}$$

Similarly, no central extension of GCA_3 or GCA_5 can be metric because

$$\langle V_0, V_0\rangle = \langle V_0, [S_+, V_-]\rangle = \langle [V_0, S_+], V_-\rangle = 0. \tag{158}$$

No central extension of GCA_2 can be metric:

$$\langle S_+, S_-\rangle = \langle S_+, [V_0, V_-]\rangle = \langle [S_+, V_0], V_-\rangle = 0, \tag{159}$$

and neither can any central extension of GCA_6:

$$\langle S_+, S_-\rangle = \langle S_+, [V_0, V_-]\rangle = \langle [V_-, S_+], V_0\rangle = 0, \tag{160}$$

or of GCA_9:

$$\langle S_+, S_-\rangle = \langle [V_0, V_+], S_-\rangle = \langle V_0, [V_+, S_0]\rangle = 0. \tag{161}$$

This leaves only GCA_1 and GCA_8 to consider. We will see that in each case there is a quotient of the universal central extension which is metric. We will do this by exhibiting them as suitable double extensions [15, 16].

Let $\mathfrak{a}$ denote the abelian Lie algebra spanned by $S_\pm, V_\pm, V_0$ and with inner product

$$\langle S_+, S_-\rangle = \alpha, \qquad \langle V_{+a}, V_{-b}\rangle = \beta\delta_{ab} \qquad \text{and} \qquad \langle V_{0a}, V_{0b}\rangle = \gamma\delta_{ab}, \tag{162}$$

for some $\alpha, \beta, \gamma \in \mathbb{R}^\times$. Then J and D act on $\mathfrak{a}$ via (3) preserving the inner product. Introduce central elements Z and Z' dual to J and D, respectively, so that

$$\langle Z, J\rangle = 1 \qquad \text{and} \qquad \langle Z', D\rangle = 1, \tag{163}$$

and centrally extend $\mathfrak{a}$ as follows:

$$[V_{0a}, V_{0b}] = \gamma\epsilon_{ab}Z, \qquad [V_{+a}, V_{-b}] = \beta\epsilon_{ab}Z + \beta\delta_{ab}Z' \qquad \text{and} \qquad [S_+, S_-] = \alpha Z'. \tag{164}$$

If we normalise the inner product by setting $\beta = 1$, then we see that this metric Lie algebra corresponds to the quotient of the universal central extension of GCA_1 in Table 18 where $Z_1 = 0$, $Z_2 = Z'$, $Z_3 = Z$, $Z_4 = \gamma Z$ and $Z_5 = \alpha Z'$. The most general invariant inner product (up to scale) is given by equations (162) and (163) and *any* symmetric bilinear form on the span of J and D.

Finally, let $\mathfrak{g}$ denote the Lie algebra spanned by $S_\pm, V_\pm, V_0$ subject to the brackets (omitting indices)

$$[S_+, V_-] = V_0, \qquad [S_+, V_0] = V_+ \qquad \text{and} \qquad [V_0, V_-] = S_-. \tag{165}$$

The Lie algebra $\mathfrak{g}$ is metric relative to the following inner product

$$\langle S_+, S_-\rangle = 1, \qquad \langle V_{0a}, V_{0b}\rangle = -\delta_{ab} \qquad \text{and} \qquad \langle V_{+a}, V_{+b}\rangle = \delta_{ab}. \tag{166}$$

The generators J and D act on $\mathfrak{g}$ via (3) preserving both the brackets and the inner product. Therefore we can introduce dual generators Z, Z' to J and D, respectively, with the corresponding dual pairing

$$\langle J, Z\rangle = 1 \qquad \text{and} \qquad \langle D, Z'\rangle = 1, \tag{167}$$

and centrally extend $\mathfrak{g}$ via

$$[V_{0a}, V_{0b}] = -\epsilon_{ab}Z, \qquad [V_{+a}, V_{-b}] = \epsilon_{ab}Z + \delta_{ab}Z' \qquad \text{and} \qquad [S_+, S_-] = Z'. \tag{168}$$

The resulting Lie algebra is metric relative to the inner product given by (166) and (167) and *any* symmetric bilinear form in the span of J and D. Comparing with Table 18, we see that this is the quotient of the universal central extension of GCA_8 where $Z_1 = 0$, $Z_2 = Z'$ and $Z_3 = Z$.

6. Generalised conformal algebras

A more general notion of conformal algebra results from dropping the requirement that the Lie algebra be graded by the action of D. In other words, a modified definition of conformal algebra might be the following:

Definition 2. By a **generalised conformal Lie algebra** with d-dimensional space isotropy we mean a real $\frac{1}{2}(d+2)(d+3)$-dimensional Lie algebra with generators $J_{ab} = -J_{ba}$, with $1 \leqslant a, b \leqslant d$, spanning a Lie subalgebra $\mathfrak{r} \cong \mathfrak{so}(d)$; that is,

$$[J_{ab}, J_{cd}] = \delta_{bc}J_{ad} - \delta_{ac}J_{bd} - \delta_{bd}J_{ac} + \delta_{ad}J_{bc}, \tag{169}$$

and $3d+3$ generators: rotational vectors V^i_a, for $i = 1, 2, 3$, and rotational scalars S^A, for $A = 1, 2, 3$. In other words, under $\mathfrak{so}(d)$ they transform like

$$[J_{ab}, V^i_c] = \delta_{bc}V^i_a - \delta_{ac}V^i_b \qquad \text{and} \qquad [J_{ab}, S^A] = 0. \tag{170}$$

The rest of the brackets between V^i_a and S^I are only subject to the Jacobi identity: in particular, they must be $\mathfrak{r}$-equivariant.

In this definition we have dropped the condition that D is a grading element. It is not clear why one should call such an algebra conformal, since all it shares with the simple conformal algebras is the existence of an $\mathfrak{so}(d)$ subalgebra which acts in the same way on the additional generators. But these algebras form a large class of Lie algebras with relations to the other candidates for the conformal fellowship.

The classification problem for generalised conformal algebras is much more involved than for the graded conformal algebras treated in the bulk of this paper and we will not solve it here. Nevertheless let us make a few comments. If $d = 0$, a conformal Lie algebra has dimension 3 and hence this agrees with the Bianchi classification of real three-dimensional Lie algebras [17] (see [18] for an English translation). In contrast, the graded conformal algebras for $d = 0$ are Bianchi VI_0 and Bianchi VIII. If $d = 1$ there are no rotations and any real Lie algebra of dimension 6 is conformal. Six-dimensional real Lie algebras have been classified in a series of papers starting with Morozov [19], who classified nilpotent six-dimensional Lie algebras, then Mubarakzjanov [20], who classified solvable six-dimensional solvable Lie algebras with five-dimensional nil-radical and Turkowski [21], who classified six-dimensional solvable Lie algebras with a four-dimensional nil-radical. Having thus classified the solvable six-dimensional Lie algebras, it remains to classify the non-solvable ones, which can be done in principle via the Levi decomposition, although I am not aware of any published list. The case of the graded conformal algebras for $d = 1$ has not been studied.

6.1. **The deformation complex for** $d \geqslant 5$**.** In this section we will take $d \geqslant 5$; that being the generic range of dimensions. As before, the deformation theory approach to the classification of generalised conformal Lie algebras consists in a perturbative approach to the solution of the corresponding Maurer–Cartan equation for the difference between the Lie brackets of the Lie algebras we want to classify and the ones of the static Lie algebra, here denoted $\mathfrak{g}$. This approach consists in the first instance in the calculation of the relative Lie algebra cohomology group $H^2(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$, with $\mathfrak{r}$ the rotational subalgebra, which classifies the infinitesimal deformations. The obstructions to integrating infinitesimal deformations live in $H^3(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$; although it is seldom necessary or convenient to calculate that cohomology group. What we will need to compute is the Nijenhuis–Richardson bracket on the cochains

$$[\![-,-]\!] : C^2(\mathfrak{g}, \mathfrak{r}; \mathfrak{g}) \times C^2(\mathfrak{g}, \mathfrak{r}; \mathfrak{g}) \to C^3(\mathfrak{g}, \mathfrak{r}; \mathfrak{g}), \tag{171}$$

which is symmetric in this degree.

In this section we will determine the first few spaces $C^\bullet(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$ of cochains, the action of the differential and the Nijenhuis–Richardson bracket.

We start by introducing a convenient notation to perform the necessary calculations. We will let V^i_a, for $i = 1, 2, 3$, stand for B_a, P_a and K_a, respectively. Similarly we will let S^A, for $A = 1, 2, 3$, stand for D, H and L, respectively. In this notation, the static conformal Lie algebra $\mathfrak{g}$ is the real span of J_{ab}, V^i_a and S^A and subject to the following brackets:

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \delta_{bc} J_{ad} - \delta_{ac} J_{bd} - \delta_{bd} J_{ac} + \delta_{ad} J_{bc} \\
[J_{ab}, V^i_c] &= \delta_{bc} V^i_a - \delta_{ac} V^i_b \\
[J_{ab}, S^A] &= 0,
\end{aligned} \tag{172}$$

with all other brackets vanishing.

We will let $\mathbb{W}$ denote the real vector space spanned by the V^i_a and the S^A. The canonical dual basis for $\mathbb{W}^*$ is denoted υ_{ai} and σ_A, respectively. We may use the rotational invariant δ_{ab} to raise and lower the vector $\mathfrak{so}(d)$ indices with impunity, but the automorphism group of the deformation complex, which is isomorphic to $GL(3, \mathbb{R}) \times GL(3, \mathbb{R})$ acting in the natural way on the three copies of the vector representation of $\mathfrak{so}(d)$ and on the three copies of the scalar representation of $\mathfrak{so}(d)$, does distinguish between these representations and their duals, hence the need to be careful with where we position the indices.

The cochain complex $C^\bullet := C^\bullet(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$ is isomorphic to $(\Lambda^\bullet \mathbb{W}^* \otimes \mathfrak{g})^{\mathfrak{r}}$ and we proceed to enumerate the first few spaces. Omitting the $\wedge$ and $\otimes$ products, we see that

(0) C^0 is spanned by the three cochains S^A, for $A = 1, 2, 3$;
(1) C^1 is spanned by the $18 = 9 + 9$ cochains $\sigma_A S^B$ and $\upsilon_{ai} V^j_a$, where we are summing over a in order to arrive at a rotational scalar;
(2) C^2 is spanned by the $51 = 9 + 27 + 6 + 9$ cochains $\sigma_A \sigma_B S^C$, $\sigma_A \upsilon_{ai} V^j_a$, $\upsilon_{ai} \upsilon_{bj} J_{ab}$ and $\upsilon_{ai} \upsilon_{aj} S^A$; and
(3) C^3 is spanned by the $102 = 3+27+18+27+27$ cochains $\sigma_A \sigma_B \sigma_C S^D$, $\sigma_A \sigma_B \upsilon_{ai} V^j_a$, $\sigma_A \upsilon_{ai} \upsilon_{bj} J_{ab}$, $\sigma_A \upsilon_{ai} \upsilon_{aj} S^B$ and $\upsilon_{ai} \upsilon_{aj} \upsilon_{bk} V^\ell_b$.

The differential $\partial : C^\bullet \to C^{\bullet+1}$ is uniquely defined by its action on the generators:

$$\partial S^A = \partial V^i_a = \partial \sigma_A = \partial \upsilon_{ai} = 0 \qquad \text{and} \qquad \partial J_{ab} = \upsilon_{ai} V^i_b - \upsilon_{bi} V^i_a. \tag{173}$$

6.2. **Infinitesimal deformations.** It follows from the above expression for the differential that $\partial : C^0 \to C^1$ and $\partial : C^1 \to C^2$ are the zero maps and thus the space of infinitesimal deformations is naturally isomorphic to the kernel of $\partial : C^2 \to C^3$. Computing the differential on the above basis for C^2 we find

$$\partial(\sigma_A \sigma_B S^C) = 0 \qquad \partial(\sigma_A \upsilon_{ai} V^j_a) = 0 \qquad \partial(\upsilon_{ai} \upsilon_{bj} J_{ab}) = \upsilon_{ak} (\upsilon_{ai} \upsilon_{bj} + \upsilon_{aj} \upsilon_{bi}) V^k_b \qquad \partial(\upsilon_{ai} \upsilon_{aj} S^A) = 0. \tag{174}$$

Therefore $H^2(\mathfrak{g}, \mathfrak{r}; \mathfrak{g})$ is a real 45-dimensional vector space spanned by

$$\sigma_A \sigma_B S^C, \qquad \sigma_A \upsilon_{ai} V^j_a \qquad \text{and} \qquad \upsilon_{ai} \upsilon_{aj} S^A. \tag{175}$$

The general infinitesimal deformation is given by

$$\varphi_1 = t^{AB}_C \sigma_A \sigma_B S^C + t^{A\,i}{}_j \sigma_A \upsilon_{ai} V^j_a + t^{ij}_A \upsilon_{ai} \upsilon_{aj} S^A, \tag{176}$$

for real parameters $t^{AB}_C = -t^{BA}_C$, $t^{A\,i}{}_j$ and $t^{ij}_A = -t^{ji}_A$.

6.3. **Obstructions to integrability.** In order to explore the integrability properties of the infinitesimal deformations we will need to calculate the Nijenhuis–Richardson bracket on C^2. This is tabulated below in Table 19, where we have abbreviated the notation by omitting the $\wedge$ and $\otimes$ products.

Table 19. Nijenhuis–Richardson $\bullet$

$\bullet$	$\sigma_D\sigma_E S^F$	$\sigma_D \upsilon_{ck} V_c^\ell$	$\upsilon_{ck}\upsilon_{d\ell} J_{cd}$	$\upsilon_{ck}\upsilon_{c\ell} S^D$
$\sigma_A\sigma_B S^C$	$\sigma_A\sigma_B(\delta_D^C\sigma_E - \delta_E^C\sigma_D)S^F$	$\delta_D^C\sigma_A\sigma_B\upsilon_{ck}V_c^\ell$	0	0
$\sigma_A\upsilon_{ai}V_a^j$	0	$\delta_k^j\sigma_A\sigma_D\upsilon_{ai}V_a^\ell$	$\sigma_A\upsilon_{ai}(\delta_k^j\upsilon_{c\ell} + \delta_\ell^j\upsilon_{ck})J_{ac}$	$\sigma_A\upsilon_{ai}(\delta_k^j\upsilon_{a\ell} - \delta_\ell^j\upsilon_{ak})S^D$
$\upsilon_{ai}\upsilon_{bj}J_{ab}$	0	0	0	0
$\upsilon_{ai}\upsilon_{bj}S^A$	$\upsilon_{ai}\upsilon_{aj}(\delta_D^A\sigma_E - \delta_E^A\sigma_D)S^F$	$\delta_D^A\upsilon_{ai}\upsilon_{aj}\upsilon_{ck}V_c^\ell$	0	0

From Table 19 it is easy to write down the Nijenhuis–Richardson bracket, which in this degree is the anti-commutator of the (non-associative) $\bullet$ product. The nonzero brackets are given by

$$\begin{aligned}
[\![\sigma_A\sigma_B S^C, \sigma_D\sigma_E S^F]\!] &= \sigma_A\sigma_B(\delta_D^C\sigma_E - \delta_E^C\sigma_D)S^F + \sigma_D\sigma_E(\delta_A^F\sigma_B - \delta_B^F\sigma_A)S^C\\
[\![\sigma_A\sigma_B S^C, \sigma_D\upsilon_{ck}V_c^\ell]\!] &= \delta_D^C\sigma_A\sigma_B\upsilon_{ck}V_c^\ell\\
[\![\sigma_A\sigma_B S^C, \upsilon_{ck}\upsilon_{c\ell}S^D]\!] &= \upsilon_{ck}\upsilon_{c\ell}(\delta_A^D\sigma_B - \delta_B^D\sigma_A)S^C\\
[\![\sigma_A\upsilon_{ai}V_a^j, \sigma_D\upsilon_{ck}V_c^\ell]\!] &= \sigma_A\sigma_D(\delta_k^j\upsilon_{ai}V_a^\ell - \delta_i^\ell\upsilon_{ak}V_a^j)\\
[\![\sigma_A\upsilon_{ai}V_a^j, \upsilon_{ck}\upsilon_{d\ell}J_{cd}]\!] &= \sigma_A\upsilon_{ai}(\delta_k^j\upsilon_{c\ell} + \delta_\ell^j\upsilon_{ck})J_{ac}\\
[\![\sigma_A\upsilon_{ai}V_a^j, \upsilon_{ck}\upsilon_{c\ell}S^D]\!] &= \sigma_A\upsilon_{ai}(\delta_k^j\upsilon_{a\ell} - \delta_\ell^j\upsilon_{ak})S^D + \delta_A^D\upsilon_{ck}\upsilon_{c\ell}\upsilon_{ai}V_a^j
\end{aligned} \tag{177}$$

Let φ_1 be the general infinitesimal deformation in equation (176). The first obstruction to integrating φ_1 is the cohomology class of $\frac12[\![\varphi_1,\varphi_1]\!]$ in $H^3(\mathfrak{g},\mathfrak{r};\mathfrak{g})$. From equation (177) we calculate

$$\begin{split}
\tfrac12[\![\varphi_1,\varphi_1]\!] = 2t_C^{AB}t_F^{CE}\sigma_A\sigma_B\sigma_E S^F + \left(t_C^{AB}t^{C\,k}_{\ \ell} + \tfrac12 t^{A\,k}_{\ j}t^{B\,j}_{\ \ell} - \tfrac12 t^{B\,k}_{\ j}t^{A\,j}_{\ \ell}\right)\sigma_A\sigma_B\upsilon_{ak}V_a^\ell\\
+ 2\left(t_C^{AB}t_A^{k\ell} + \tfrac12 t^{B\,k}_{\ i}t_C^{i\ell} - \tfrac12 t^{B\,\ell}_{\ i}t_C^{ik}\right)\sigma_B\upsilon_{bk}\upsilon_{b\ell}S^C + t^{A\,i}_{\ j}t_A^{k\ell}\upsilon_{ai}\upsilon_{bk}\upsilon_{b\ell}V_a^j
\end{split} \tag{178}$$

From equation (174), the only way that this can be $\partial\varphi_2$ for some $\varphi_2 \in C^2$, is if the following equations are satisfied:

$$\begin{aligned}
t_C^{[AB}t_F^{E]C} &= 0\\
t^{A\,k}_{\ j}t^{B\,j}_{\ \ell} - t^{B\,k}_{\ j}t^{A\,j}_{\ \ell} &= -2t_C^{AB}t^{C\,k}_{\ \ell}\\
t^{B\,k}_{\ i}t_C^{i\ell} - t^{B\,\ell}_{\ i}t_C^{ik} &= -2t_C^{AB}t_A^{k\ell}\\
2t^{A\,i}_{\ j}t_A^{k\ell} &= u^{\ell i}\delta_j^k - u^{ki}\delta_j^\ell \qquad \exists u^{ij} = u^{ji},
\end{aligned} \tag{179}$$

where $\varphi_2 = \frac12 u^{ij}\upsilon_{ai}\upsilon_{bj}J_{ab}$. Assuming these equations, the next obstruction is the class in $H^3(\mathfrak{g},\mathfrak{r};\mathfrak{g})$ of

$$[\![\varphi_1,\varphi_2]\!] = t^{A\,i}_{\ j}u^{j\ell}\sigma_A\upsilon_{ai}\upsilon_{c\ell}J_{ac}. \tag{180}$$

The only way that this can be a coboundary is if it vanishes identically, which becomes the second-order obstruction

$$u^{j\ell}t^{A\,i}_{\ j} + u^{ji}t^{A\,\ell}_{\ j} = 0. \tag{181}$$

If this is the case, then we can take $\varphi_3 = 0$ and since $[\![\varphi_2,\varphi_2]\!] = 0$ identically, we can also take $\varphi_{n>3} = 0$ and the deformation integrates. In summary, the integrability domain is the locus of the equations (179) and (181).

Contracting the last equation in (179) with δ_k^j, we can solve

$$u^{\ell i} = t^{A\,i}_{\ k}t_A^{k\ell}, \tag{182}$$

which implies for consistency the symmetry in $\ell \leftrightarrow i$ of the RHS:

$$t^{A\,i}_{\ k}t_A^{k\ell} = t^{A\,\ell}_{\ k}t_A^{ki}. \tag{183}$$

In summary, the most general deformation is given by

$$\varphi = t_C^{AB}\sigma_A\sigma_B S^C + t^{A\,i}_{\ j}\sigma_A\upsilon_{ai}V_a^j + t_A^{ij}\upsilon_{ai}\upsilon_{aj}S^A + \tfrac12 t^{A\,j}_{\ k}t_A^{ki}\upsilon_{ai}\upsilon_{bj}J_{ab}, \tag{184}$$

for some parameters t^{AB}_C, $t^{A}{}^{i}_{j}$ and t^{ij}_A and where we have omitted $\otimes$ and $\wedge$. This deformation is integrable provided the following polynomial equations of second and third degree are satisfied:

$$\begin{aligned} t^{[AB}_C t^{E]C}_F &= 0 \\ t^{A}{}^{k}_{j} t^{B}{}^{j}_{\ell} - t^{B}{}^{k}_{j} t^{A}{}^{j}_{\ell} &= -2 t^{AB}_C t^{C}{}^{k}_{\ell} \\ t^{B}{}^{k}_{i} t^{i\ell}_C - t^{B}{}^{\ell}_{i} t^{ik}_C &= -2 t^{AB}_C t^{k\ell}_A \\ t^{A}{}^{i}_{j} t^{j\ell}_A &= t^{A}{}^{\ell}_{j} t^{ji}_A \\ 2 t^{A}{}^{i}_{j} t^{k\ell}_A &= t^{A}{}^{i}_{m} t^{m\ell}_A \delta^k_j - t^{A}{}^{i}_{m} t^{mk}_A \delta^\ell_j \\ t^{A}{}^{i}_{j} t^{B}{}^{j}_{k} t^{k\ell}_B + t^{A}{}^{\ell}_{j} t^{B}{}^{j}_{k} t^{ki}_B &= 0. \end{aligned} \tag{185}$$

The integrability locus is the common zero of these equations and the moduli space of conformal algebras (for $d \geqslant 4$) is the quotient of the integrability locus by the natural action of $GL(3, \mathbb{R}) \times GL(3, \mathbb{R})$ on the lower and upper case indices, respectively.

Some of the equations in (185) admit a natural Lie-theoretical interpretation. For example, the first equation says that t^{AB}_C are the structure constants of a real three-dimensional Lie algebra $\mathfrak{b}$. The second equation says that $-\frac{1}{2} t^{A}{}^{i}_{j}$ define a three-dimensional real representation E, say, of $\mathfrak{b}$. The third equation says that t^{ij}_A defines a $\mathfrak{b}$-equivariant map $\Lambda^2 E \to \mathfrak{b}$. The fourth equation can be rewritten as

$$0 = t^{A}{}^{i}_{j} t^{j\ell}_A - t^{A}{}^{\ell}_{j} t^{ji}_A = \delta^A_B \left(t^{B}{}^{i}_{j} t^{j\ell}_A - t^{B}{}^{\ell}_{j} t^{ji}_A \right) = \delta^A_B \left(t^{B}{}^{i}_{j} t^{j\ell}_A + t^{B}{}^{\ell}_{j} t^{ij}_A \right) = \delta^A_B \left(2 t^{BC}_A t^{i\ell}_C \right) = 2 t^{AC}_A t^{i\ell}_C, \tag{186}$$

where in the penultimate equality we used the equivariance of t^{ij}_A under $\mathfrak{b}$. Notice that this equation is identically zero if $\mathfrak{b}$ is unimodular. We therefore see that in order to solve the obstruction conditions requires, as a first step, classifying the three-dimensional representations of all the Bianchi Lie algebras.

As with deformations of kinematical and graded conformal Lie algebras, these are also the results for $d = 4$. In $d = 4$ we find that $\mathfrak{so}(4)$ is not simple, but only semisimple and hence we may split the rotational generators J_{ab} into their self-dual and anti-self-dual components. This does not alter the calculations of the infinitesimal deformations and the additional freedom it gives in the form of φ_2 is not used in overcoming the obstruction to integrability to second-order. Therefore the results above for $d \geqslant 5$ actually apply to all $d \geqslant 4$.

Finally, we end this section with the observation that holographic conformal algebras (defined in the introduction) are special cases of the generalised conformal algebras in Definition 2. Indeed, a generalised conformal algebra is holographic if and only if it contains an $\mathfrak{so}(d+1)$-subalgebra, so that as a vector space, a holographic conformal algebra $\mathfrak{g}$ decomposes as

$$\mathfrak{g} = \underbrace{\mathfrak{so}(d) \oplus \mathbb{V} \oplus}_{\cong \mathfrak{so}(d+1)} \quad \underbrace{(\mathbb{V} \oplus S)}_{\mathfrak{so}(d+1)-\text{vector}} \quad \oplus \quad \underbrace{(\mathbb{V} \oplus S)}_{\mathfrak{so}(d+1)-\text{vector}} \quad \oplus \quad \underbrace{S}_{\mathfrak{so}(d+1)-\text{scalar}} \quad . \tag{187}$$

As shown in Section 3.5, the only graded conformal algebras which are holographic are the de Sitter algebras ($\mathsf{GCA}^{(\varepsilon)}_{15}$) and the Newton–Hooke algebra ($\mathsf{GCA}^{(\varepsilon=1)}_{13}$).

7. Generalised Lifshitz algebras

We shall now consider a generalisation of the graded conformal algebras which results from demanding the existence of a grading, but not necessarily with the same weights as in Definition 1. This generalises the Lifshitz algebra extended by boosts (see, e.g., [22]).

Definition 3. A **generalised Lifshitz algebra** (with d-dimensional space isotropy) is a real Lie algebra $\mathfrak{g}$ of dimension $\frac{1}{2}(d+1)(d+2)+1$ satisfying the following properties:

(1) $\mathfrak{g}$ has a Lie subalgebra $\mathfrak{h} \cong \mathfrak{co}(d)$, and
(2) as a vector space, $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{V}_\alpha \oplus \mathbb{V}_\beta \oplus S_\gamma$, where $\mathbb{V}_{\alpha,\beta}$ are copies of the d-dimensional vector representation of $\mathfrak{so}(d)$ with weights α, β relative to D and S_γ is a copy of the scalar representation of $\mathfrak{so}(d)$ and weight γ relative to D.

It follows from the definition, that a generalised Lifshitz algebra is nothing but a kinematical Lie algebra $\mathfrak{k}$ (with d-dimensional space isotropy) extended by a grading element D which commutes with the rotations. Therefore to classify generalised Lifshitz algebras we need only classify possible gradings of kinematical Lie algebras where $\mathfrak{so}(d)$ lies in degree 0. Every such grading defines a derivation on $\mathfrak{k}$ which commutes with the rotations and which is diagonalisable over $\mathbb{R}$. Every such derivation integrates to an automorphism of $\mathfrak{k}$ which acts like the identity on the rotational subalgebra and lies in the identity component of the group of automorphisms. Automorphisms of kinematical Lie algebras have been discussed (except for a few examples where it was not then necessary) in [9] in the process of classifying simply-connected homogeneous kinematical spacetimes. It is then a matter mostly of recontextualising those calculations in [9] to arrive at the results summarised in Table 21.

In the notation of Table 21 we do not write the $\mathfrak{so}(d)$ indices explicitly. We write $\mathbf{J}$, $\mathbf{B}$, $\mathbf{P}$ and H for the generators of the kinematical Lie algebra $\mathfrak{k}$ and write the kinematical Lie brackets as

$$[\mathbf{J},\mathbf{J}] = \mathbf{J} \qquad [\mathbf{J},\mathbf{B}] = \mathbf{B} \qquad [\mathbf{J},\mathbf{P}] = \mathbf{P} \qquad \text{and} \qquad [\mathbf{J},H] = 0. \tag{188}$$

For $d \neq 2$, any other brackets can be reconstructed unambiguously from the abbreviated expression since there is only one way to reintroduce indices using only the $\mathfrak{so}(d)$-invariant tensor δ_{ab} and, when $d = 3$ also ϵ_{abc} on the right hand side of the brackets. For example,

$$[H,\mathbf{B}] = \mathbf{P} \quad \text{stands for} \quad [H,B_a] = P_a \qquad \text{and} \qquad [\mathbf{B},\mathbf{P}] = H + \mathbf{J} \quad \text{for} \quad [B_a,P_b] = \delta_{ab}H + J_{ab}. \tag{189}$$

In $d = 3$ we may also have brackets of the form

$$[\mathbf{P},\mathbf{P}] = \mathbf{P} \qquad \text{which we take to mean} \qquad [P_a,P_b] = \epsilon_{abc}P_c. \tag{190}$$

If $d = 2$, then ϵ_{ab} is rotationally invariant and can appear in Lie brackets. So we will write, e.g.,

$$[H,\mathbf{B}] = \mathbf{B} + \widetilde{\mathbf{P}} \qquad \text{for} \qquad [H,B_a] = B_a + \epsilon_{ab}P_b, \tag{191}$$

and

$$[\mathbf{B},\mathbf{B}] = \widetilde{H} \qquad \text{for} \qquad [B_a,B_b] = \epsilon_{ab}H, \tag{192}$$

et cetera. Also, whenever $\mathbf{J}$ appears it denotes J_{ab}. In $d = 1$, it is tacitly assumed that we set any $\mathbf{J}$ to zero.

In Table 21 we write the isomorphism class of the Lie algebra (if known) using the notation in Table 20.

Table 20. Notation for Lie algebras

Notation	Name
$\mathfrak{s}$	static
$\mathfrak{n}_+$	(elliptic) Newton
$\mathfrak{n}_-$	(hyperbolic) Newton
$\mathfrak{e}$	euclidean

Notation	Name
$\mathfrak{p}$	Poincaré
$\mathfrak{so}$	orthogonal
$\mathfrak{g}$	galilean
$\mathfrak{c}$	Carroll

Table 21 is subdivided into three sections separated by horizontal rules: kinematical Lie algebras which exist for generic d, those which are unique to $d = 3$ and those which are unique to $d = 2$. In $d = 1$ there are accidental isomorphisms: $\mathfrak{c} \cong \mathfrak{g}$, $\mathfrak{so}(2,1) \cong \mathfrak{so}(1,2)$, $\mathfrak{e} \cong \mathfrak{n}_+$ and $\mathfrak{p} \cong \mathfrak{n}_-$. In addition, for $d = 2$, $\mathfrak{n}_+$ admits more gradings than for $d \neq 2$, which is why we have listed it separately in $d = 2$. The notation for the grading is such that w_X denotes the degree (or "weight") of the generator X and $\alpha, \beta, \gamma \in \mathbb{R}$.

Table 21. Generalised Lifshitz algebras with d-dimensional space isotropy

GLA#	d	$\cong$	Nonzero Lie brackets in addition to $[\mathbf{J},\mathbf{J}] = \mathbf{J}$, $[\mathbf{J},\mathbf{B}] = \mathbf{B}$ and $[\mathbf{J},\mathbf{P}] = \mathbf{P}$					$w_\mathbf{B}$	$w_\mathbf{P}$	w_H
GLA_1	$\geqslant 0$	$\mathfrak{s}$						α	β	γ
GLA_2	$\geqslant 1$	$\mathfrak{g}$	$[H,\mathbf{B}] = -\mathbf{P}$					α	$\alpha+\gamma$	γ
$\mathsf{GLA}_3^{(\chi\in(-1,1))}$	$\geqslant 1$		$[H,\mathbf{B}] = \chi\mathbf{B}$	$[H,\mathbf{P}] = \mathbf{P}$				α	β	0
GLA_4	$\geqslant 1$		$[H,\mathbf{B}] = \mathbf{B}$	$[H,\mathbf{P}] = \mathbf{P}$				α	β	0
GLA_5	$\geqslant 1$	$\mathfrak{n}_-$	$[H,\mathbf{B}] = -\mathbf{B}$	$[H,\mathbf{P}] = \mathbf{P}$				α	β	0
GLA_6	$\geqslant 1$		$[H,\mathbf{B}] = -\mathbf{P}$	$[H,\mathbf{P}] = \mathbf{B} + 2\mathbf{P}$				α	α	0
$\mathsf{GLA}_7^{(\chi>0)}$	$\geqslant 1$		$[H,\mathbf{B}] = \chi\mathbf{B} + \mathbf{P}$	$[H,\mathbf{P}] = \chi\mathbf{P} - \mathbf{B}$				α	α	0
GLA_8	$\geqslant 1 (\neq 2)$	$\mathfrak{n}_+$	$[H,\mathbf{B}] = \mathbf{P}$	$[H,\mathbf{P}] = -\mathbf{B}$				α	α	0
GLA_9	$\geqslant 2$	$\mathfrak{c}$				$[\mathbf{B},\mathbf{P}] = H$		α	β	$\alpha+\beta$
$\mathsf{GLA}_{10}^{(\varepsilon=\pm1)}$	$\geqslant 2$	$\mathfrak{p}$ $\mathfrak{e}$	$[H,\mathbf{B}] = -\varepsilon\mathbf{P}$		$[\mathbf{B},\mathbf{B}] = \varepsilon\mathbf{J}$	$[\mathbf{B},\mathbf{P}] = H$		0	β	β
GLA_{11}	$\geqslant 2$	$\mathfrak{so}(d+1,1)$	$[H,\mathbf{B}] = \mathbf{B}$	$[H,\mathbf{P}] = -\mathbf{P}$		$[\mathbf{B},\mathbf{P}] = H + \mathbf{J}$		α	$-\alpha$	0
$\mathsf{GLA}_{12}^{(\varepsilon=\pm1)}$	$\geqslant 1$	$\mathfrak{so}(d,2)$ $\mathfrak{so}(d+2)$	$[H,\mathbf{B}] = -\varepsilon\mathbf{P}$	$[H,\mathbf{P}] = \varepsilon\mathbf{B}$	$[\mathbf{B},\mathbf{B}] = \varepsilon\mathbf{J}$	$[\mathbf{B},\mathbf{P}] = H$	$[\mathbf{P},\mathbf{P}] = \varepsilon\mathbf{J}$	0	0	0
$\mathsf{GLA}_{13}^{(\varepsilon=\pm1)}$	3				$[\mathbf{B},\mathbf{B}] = \mathbf{B}$		$[\mathbf{P},\mathbf{P}] = \varepsilon(\mathbf{B} - \mathbf{J})$	0	0	γ
GLA_{14}	3				$[\mathbf{B},\mathbf{B}] = \mathbf{B}$			0	β	γ
GLA_{15}	3				$[\mathbf{B},\mathbf{B}] = \mathbf{P}$			α	2α	γ
GLA_{16}	3			$[H,\mathbf{P}] = \mathbf{P}$	$[\mathbf{B},\mathbf{B}] = \mathbf{B}$			0	β	0
GLA_{17}	3		$[H,\mathbf{B}] = -\mathbf{P}$		$[\mathbf{B},\mathbf{B}] = \mathbf{P}$			α	2α	α
GLA_{18}	3		$[H,\mathbf{B}] = \mathbf{B}$	$[H,\mathbf{P}] = 2\mathbf{P}$	$[\mathbf{B},\mathbf{B}] = \mathbf{P}$			α	2α	0
$\mathsf{GLA}_{19}^{(\xi,\chi)}$	2		$[H,\mathbf{B}] = \mathbf{B}$	$[H,\mathbf{P}] = \xi\mathbf{P} - \chi\widetilde{\mathbf{P}}$		$(\xi \in [-1,1),\ \chi > 0)$		α	β	0
GLA_{20}	2	$\mathfrak{n}_+$	$[H,\mathbf{B}] = \widetilde{\mathbf{B}}$					α	β	0
GLA_{21}	2				$[\mathbf{B},\mathbf{B}] = \widetilde{H}$		$[\mathbf{P},\mathbf{P}] = \widetilde{H}$	α	α	2α
GLA_{22}	2		$[H,\mathbf{B}] = \widetilde{\mathbf{B}}$		$[\mathbf{B},\mathbf{B}] = \widetilde{H}$		$[\mathbf{P},\mathbf{P}] = \mathbf{J} + \widetilde{H}$	0	0	0
GLA_{23}	2				$[\mathbf{B},\mathbf{B}] = \widetilde{H}$			α	β	2α
GLA_{24}	2		$[H,\mathbf{B}] = \mathbf{P}$		$[\mathbf{B},\mathbf{B}] = \widetilde{H}$			α	3α	2α
$\mathsf{GLA}_{25}^{(\varepsilon=\pm1)}$	2		$[H,\mathbf{B}] = \varepsilon\widetilde{\mathbf{B}}$		$[\mathbf{B},\mathbf{B}] = \widetilde{H}$			0	β	0

To each row in Table 21 there corresponds a generalised Lifshitz algebra with additional generator D satisfying:

$$[D, \mathbf{J}] = 0, \qquad [D, \mathbf{B}] = w_{\mathbf{B}}\mathbf{B}, \qquad [D, \mathbf{P}] = w_{\mathbf{P}}\mathbf{P} \qquad \text{and} \qquad [D, H] = w_H H, \tag{193}$$

with the given $w_{\mathbf{B}}$, $w_{\mathbf{P}}$ and w_H. We may rescale D so that one of the nonzero $w_{\mathbf{B}}$, $w_{\mathbf{P}}$ and w_H is equal to 1.

In a generalised Lifshitz algebra, the grading element is an (outer) derivation of a kinematical Lie algebra. To be able to interpret the grading element as a dilatation, it seems reasonable to require that the generalised Lifshitz algebra $\mathfrak{g}$ can be "extended" (*not* in the algebraic sense of an extension) by additional generators: namely, one $\mathfrak{so}(d)$ scalar and/or one $\mathfrak{so}(d)$ vector, in such a way that the resulting algebra is still graded by the adjoint action of D, but such that now D appears in the right-hand side of a Lie bracket. This is an interesting problem which is beyond the scope of this paper.

8. Generalised Schrödinger algebras

As advocated in several places [23, 24, 14], it is worthwhile to think of the Schrödinger algebra as (the central extension of) a conformal algebra. The defining feature of Lie algebras such as the Schrödinger algebra is that it should be the central extension of a Lie algebra possessing an $\mathfrak{so}(d) \oplus \mathfrak{sl}(2,\mathbb{R})$ subalgebra and, in addition, two (or perhaps three) copies of the $\mathfrak{so}(d)$ vector representation. In other words, comparing to the generalised conformal algebra in Definition 2, one of the vector representations might be missing (typically the one which can be interpreted as spatial special conformal transformations) and the Bianchi Lie algebra spanned by the three scalars is isomorphic to Bianchi VIII. Of the graded conformal algebras classified in this paper, the only ones where the scalars span an $\mathfrak{sl}(2,\mathbb{R})$ subalgebra are the galilean conformal algebra (GCA_{14}) and the simple conformal algebras ($\mathsf{GCA}_{15}^{(\varepsilon)}$).

Definition 4. A **generalised Schrödinger algebra** (with d-dimensional space isotropy) is a real Lie algebra $\mathfrak{g}$ of dimension $\frac{1}{2}d(d+3)+4$ satisfying the following properties:

(1) $\mathfrak{g}$ has a Lie subalgebra $\mathfrak{h} \cong \mathfrak{so}(d) \oplus \mathfrak{sl}(2,\mathbb{R})$, and
(2) as a vector space, $\mathfrak{g} = \mathfrak{h} \oplus (\mathbb{V} \otimes E) \oplus \mathbb{R}Z$, where $\mathbb{V}$ is a copy of the d-dimensional vector representation of $\mathfrak{so}(d)$, E is a representation of $\mathfrak{sl}(2,\mathbb{R})$ of dimension 2 or 3, and Z is a central element.

8.1. **The case** $\dim E = 2$**.** We shall start with the case where E is two-dimensional. Observe that $\mathfrak{sl}(2,\mathbb{R})$ has precisely two inequivalent two-dimensional representations: the fundamental representation and the trivial representation. This gives rise to two classes of generalised Schrödinger algebras with $\dim E = 2$.

In more concrete terms, a generalised Schrödinger algebra (with $\dim E = 2$) admits a basis (J_{ab}, S^A, V^i_a, Z), where J_{ab} span an $\mathfrak{so}(d)$ subalgebra, S^A are rotational scalars spanning an $\mathfrak{sl}(2,\mathbb{R})$ subalgebra, V^i_a, for $i = 1, 2$, are rotational vectors and the rotational scalar Z is central. The (potentially) nonzero brackets (for $d \geqslant 4$) are given by

$$\begin{aligned}
[J_{ab}, J_{cd}] &= \delta_{bc}J_{ad} - \delta_{ac}J_{bd} - \delta_{bd}J_{ac} + \delta_{ad}J_{bc}\\
[J_{ab}, V^i_c] &= \delta_{bc}V^i_a - \delta_{ac}V^i_b\\
[S^A, S^B] &= f^{AB}{}_C S^C\\
[S^A, V^i_a] &= t^{A\,i}{}_j V^j_a
\end{aligned} \tag{194}$$

where $f^{AB}{}_C$ are the structure constants of $\mathfrak{sl}(2,\mathbb{R})$, and where $S^A \mapsto t^{A\,i}{}_j$ is a two-dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$, and in addition

$$[V^i_a, V^j_b] = \delta_{ab}\epsilon_{ij}Z + u^{ij}J_{ab}, \tag{195}$$

where $u^{ij} = u^{ji}$. Notice that we cannot have S^A appearing in the $[V, V]$ brackets, for that would require the existence of an $\mathfrak{sl}(2,\mathbb{R})$-equivariant map $f : \Lambda^2 E \to \mathfrak{sl}(2,\mathbb{R})$, but since $\Lambda^2 E$ is always a trivial representation (whether or not E is trivial), its image under f would have to be central in $\mathfrak{sl}(2,\mathbb{R})$ and $\mathfrak{sl}(2,\mathbb{R})$, being simple, has trivial centre.

In the first class of algebras, E is the trivial representation and hence $t^{A\,i}{}_j = 0$. Subjecting the Lie bracket (195) to the $[V, V, V]$ Jacobi identity, we find that

$$(u^{ij}\delta^k_\ell - u^{ki}\delta^j_\ell)\delta_{bc}\delta_{ae} + (u^{jk}\delta^i_\ell - u^{ij}\delta^k_\ell)\delta_{ca}\delta_{be} + (u^{ki}\delta^j_\ell - u^{jk}\delta^i_\ell)\delta_{ab}\delta_{ce} = 0, \tag{196}$$

for all $i, j, k, \ell = 1, 2$ and $a, b, c, e = 1, \dots, d$, where $d \geqslant 4$. Taking $a = b \neq c = e$, we are left with

$$u^{ki}\delta^j_\ell = u^{jk}\delta^i_\ell, \tag{197}$$

for all $i, j, k, \ell = 1, 2$. Put $i \neq \ell = j$ and we see that $u^{ki} = 0$ for all k and all i. Therefore,

$$[V^i_a, V^j_b] = \delta_{ab}\epsilon_{ij}Z. \tag{198}$$

In the second class of algebras, E is the fundamental representation of $\mathfrak{sl}(2,\mathbb{R})$. Then the Jacobi identity implies, in particular, that u^{ij} is $\mathfrak{sl}(2,\mathbb{R})$-invariant. But $u \in \odot^2 E$, which is a non-trivial irreducible representation of $\mathfrak{sl}(2,\mathbb{R})$, and hence the only invariant is $u = 0$. Therefore, the additional brackets are those in (195) with

$u^{ij} = 0$ and $t^{A\,i}{}_j$ the fundamental representation of $\mathfrak{sl}(2,\mathbb{R})$. This is (the $d \geqslant 4$ avatar of) the Schrödinger algebra of [25, 26].

The situation for $d = 3$ is slightly more involved for the first type of algebras where E is a trivial representation of $\mathfrak{sl}(2,\mathbb{R})$, but not for the second type of algebras where E is the fundamental representation. Treating the problem using the methods of deformation theory, we look to classify the deformations of the Lie algebra $\mathfrak{g}$ with all brackets zero except for those in (194). The deformation complex, by Hochschild–Serre, is the Chevalley–Eilenberg complex of $\mathfrak{g}$ with values in the adjoint representation, but relative to the semisimple subalgebra $\mathfrak{h} \cong \mathfrak{so}(3) \oplus \mathfrak{sl}(2,\mathbb{R})$. If we write $\mathfrak{g} = \mathfrak{h} \oplus \mathbb{W}$, where $\mathbb{W}$ is the span of the V^i_a, then the cochains in the deformation complex are $C^p := (\Lambda^p \mathbb{W}^* \otimes \mathfrak{g})^{\mathfrak{h}}$. The differential in this complex is zero, so $H^p = C^p$ for all p. If E is the fundamental representation of $\mathfrak{sl}(2,\mathbb{R})$, it is an easy calculation using the representation theory of $\mathfrak{so}(3)$ and $\mathfrak{sl}(2,\mathbb{R})$, that $C^2 = 0$. Therefore for E the fundamental representation, there are no deformations of $\mathfrak{g}$ and hence its central extension is again the Schrödinger algebra.

For E the trivial representation, however, equation (195) is replaced with

$$[V^i_a, V^j_b] = \delta_{ab}\epsilon_{ij} Z + u^{ij} J_{ab} + t^{ij}_k \epsilon_{abc} V^k_c, \tag{199}$$

subject to the $[V,V,V]$ Jacobi identity. There are three components to the Jacobi identity: the one along Z, the one along J and the one along V itself. The component along Z says

$$t^{ij}_\ell \epsilon^{\ell k} + t^{jk}_\ell \epsilon^{\ell i} + t^{ki}_\ell \epsilon^{\ell j} = 0, \tag{200}$$

which translates into

$$t^{11}_2 = t^{22}_1 = 0, \qquad t^{11}_1 = 2t^{12}_2 \qquad \text{and} \qquad t^{22}_2 = 2t^{12}_1. \tag{201}$$

The component along J gives

$$t^{ij}_\ell u^{\ell k}(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}) + t^{jk}_\ell u^{\ell i}(\delta_{ba}\delta_{cd} - \delta_{bd}\delta_{ca}) + t^{ki}_\ell u^{\ell j}(\delta_{cb}\delta_{ad} - \delta_{cd}\delta_{ab}) = 0, \tag{202}$$

which taking (201) into account gives two relations

$$2t^{12}_1 u^{11} = t^{11}_1 u^{12} \qquad \text{and} \qquad 2t^{12}_1 u^{12} = t^{11}_1 u^{22}. \tag{203}$$

Finally, the component of the $[V,V,V]$ Jacobi identity along V gives

$$(u^{ij}\delta^k_\ell - t^{ij}_m t^{mk}_\ell)(\delta_{bc}\delta_{ad} - \delta_{ac}\delta_{bd}) + (u^{jk}\delta^i_\ell - t^{jk}_m t^{mi}_\ell)(\delta_{ca}\delta_{bd} - \delta_{ba}\delta_{cd}) + (u^{ki}\delta^j_\ell - t^{ki}_m t^{mj}_\ell)(\delta_{ab}\delta_{cd} - \delta_{cb}\delta_{ad}) = 0, \tag{204}$$

which taking (201) into account, simply allows us to solve for the u^{ij} in terms of the t^{ij}_k:

$$u^{11} = \left(t^{12}_2\right)^2, \qquad u^{12} = t^{12}_1 t^{12}_2 \qquad \text{and} \qquad u^{22} = \left(t^{12}_1\right)^2. \tag{205}$$

Inserting these back into equation (203) we see that the two equations are identically satisfied.

Letting $t^{12}_1 =: \lambda$ and $t^{12}_2 =: \mu$, the $[V,V]$ bracket becomes (in shorthand notation)

$$[V^1, V^1] = \mu^2 J + 2\mu V^1, \qquad [V^1, V^2] = Z + \lambda\mu J + \lambda V^1 + \mu V^2 \qquad \text{and} \qquad [V^2, V^2] = \lambda^2 J + 2\lambda V^2. \tag{206}$$

We have four cases depending on whether or not λ and μ are zero:

(1) $\lambda = \mu = 0$: the only nonzero bracket is

$$[V^1_a, V^2_b] = \delta_{ab} Z; \tag{207}$$

(2) $\lambda = 0$ and $\mu \neq 0$: the nonzero brackets are now (after rescaling V^1 by μ^{-1}):

$$[V^1_a, V^1_b] = J_{ab} + 2\epsilon_{abc} V^1_c \qquad \text{and} \qquad [V^1_a, V^2_b] = \delta_{ab} Z + \epsilon_{abc} V^2_c, \tag{208}$$

but changing basis to $V^1_a \mapsto V^1_a + \frac{1}{2}\epsilon_{abc} J_{bc}$, we arrive at only one nonzero bracket: namely,

$$[V^1_a, V^2_b] = \delta_{ab} Z; \tag{209}$$

(3) $\lambda \neq 0$ and $\mu = 0$ is isomorphic to the previous case via the change of basis $(J, V^1, V^2, Z) \mapsto (J, V^2, V^1, -Z)$; and

(4) $\lambda\mu \neq 0$, where the nonzero brackets are now (after rescaling V^1 by μ^{-1} and V^2 by λ^{-1}):

$$\begin{aligned} [V^1_a, V^1_b] &= J_{ab} + 2\epsilon_{abc} V^1_c \\ [V^2_a, V^2_b] &= J_{ab} + 2\epsilon_{abc} V^2_c \\ [V^1_a, V^2_b] &= \delta_{ab} Z + J_{ab} + \epsilon_{abc}(V^1_c + V^2_c), \end{aligned} \tag{210}$$

which, changing basis to $V^\pm := \frac{1}{2}(V^1 \pm V^2)$, simplifies to the following non-zero brackets:

$$[V^+_a, V^+_b] = J_{ab} + 2\epsilon_{abc} V^+_c \qquad \text{and} \qquad [V^+_a, V^-_b] = -\tfrac{1}{2}\delta_{ab} Z + \epsilon_{abc} V^-_c, \tag{211}$$

and, finally, changing basis $V^+_a \mapsto V^+_a + \frac{1}{2}\epsilon_{abc} J_{bc}$ and redefining Z, we find that the only nonzero bracket remains

$$[V^+_a, V^-_b] = \delta_{ab} Z. \tag{212}$$

In summary, we find again that just as for $d \geqslant 4$, there is only one isomorphism class of Lie algebras, with additional bracket given by equation (198).

8.2. **The case** $\dim E = 3$**.** Consider now the case where E is a three-dimensional representation of $\mathfrak{sl}(2,\mathbb{R})$. Then we are essentially in the special case of Definition 2 where the Lie algebra spanned by the scalars is isomorphic to $\mathfrak{sl}(2,\mathbb{R})$ or, equivalently, Bianchi VIII. The representation E is isomorphic to one of the following three representations:

(1) the trivial three-dimensional representation $\mathbb{R}^3$;
(2) the direct sum $F \oplus \mathbb{R}$, where $\mathbb{R}$ is the trivial one-dimensional representation and F is the fundamental two-dimensional representation; or
(3) the adjoint representation.

Restricting to $d \geqslant 4$, we have that the most general deformation is given by equation (184) and the obstructions to integrating it are given by equation (185). The first three equations say that t_C^{AB} are the structure constants of $\mathfrak{sl}(2,\mathbb{R})$ relative to some basis, $-\frac{1}{2}t^{A}{}_j^i$ define the three-dimensional representation E and t_A^{ij} defines an $\mathfrak{sl}(2,\mathbb{R})$-equivariant linear map $\Lambda^2 E \to \mathfrak{sl}(2,\mathbb{R})$. The fourth obstruction equation vanishes because $\mathfrak{sl}(2,\mathbb{R})$ is unimodular, as shown by equation (186).

If E is the trivial representation, then $t^{A}{}_j^i = 0$ and since $\mathfrak{sl}(2,\mathbb{R})$ has no centre, $t_A^{ij} = 0$ as well. The remaining two equations in (185) are automatically satisfied. Hence the algebra is given by the common Lie brackets (169) and (170) and in addition $[S^A, S^B] = t_C^{AB} S^C$ defining $\mathfrak{sl}(2,\mathbb{R})$. We may centrally extend this algebra by generators $Z^{ij} = -Z^{ji}$ and brackets

$$[V_a^i, V_b^j] = \delta_{ab} Z^{ij}. \tag{213}$$

If $E = F \oplus \mathbb{R}^2$, then $\Lambda^2 E \cong E$ as $\mathfrak{sl}(2,\mathbb{R})$-representations and since there is no $\mathfrak{sl}(2,\mathbb{R})$-equivariant map $E \to \mathfrak{sl}(2,\mathbb{R})$, we see that $t_A^{ij} = 0$. The last two equations in (185) are automatically satisfied. Therefore the Lie algebra is spanned by $J, V^0, V^+, V^-, S^0, S^+, S^-$, with Lie brackets given by (169) and (170) and in addition

$$[S^\pm, V^\mp] = V^\pm, \quad [S^0, V^\pm] = \pm V^\pm, \quad [S^0, S^\pm] = \pm 2S^\pm \quad \text{and} \quad [S^+, S^-] = S^0. \tag{214}$$

The Lie algebra admits a central extension with generator Z and Lie brackets

$$[V_a^+, V_b^-] = \delta_{ab} Z. \tag{215}$$

Notice that the subspace spanned by V_0 is an ideal and quotienting by this ideal gives again the Schrödinger algebra. This Lie algebra is graded, but the weights are not those of a graded conformal algebra as in Definition 1. This suggests relaxing the definition of a graded conformal algebra by allowing arbitrary conformal weights. We discussed this in Section 7 in the context of generalised Lifshitz algebras.

Finally, if E is the adjoint representation, then now t_A^{ij} is a constant multiple of the isomorphism $\Lambda^2 E \to E \cong \mathfrak{sl}(2,\mathbb{R})$. For any value of this constant, one can check explicitly that the last two equations in (185) are satisfied. Therefore the resulting Lie algebra is isomorphic to one of the simple conformal algebras $\mathsf{GCA}_{15}^{(\varepsilon)}$. These algebras do not admit any nontrivial central extensions.

9. Conclusions and open problems

This paper is devoted to the slippery notion of a conformal algebra. We have focussed primarily on the notion of a graded conformal algebra with d-dimensional space isotropy (see Definition 1) and we have classified them up to isomorphism for all $d \geqslant 2$. We have done this by classifying deformations of the static graded conformal algebra defined in the introduction. For each such $d \neq 3$ there are 17 isomorphism classes, which are listed in Table 6, which also applies when $d = 2$. For $d = 3$ there are in addition another 6 isomorphism classes, which are listed in Table 11. Some of these Lie algebras are related by contraction and this defines a partial order in the set of isomorphism classes whose Hasse diagram is depicted in Figure 1, which also applies to $d = 2$. With the exception of the simple Lie algebras $\mathsf{GCA}_{15}^{(\varepsilon)}$ (isomorphic to $\mathfrak{so}(d+2,1)$ or $\mathfrak{so}(d+1,2)$, respectively), none of the other graded conformal algebras admit an invariant inner product. We then classified the central extensions of these Lie algebras. The situation was different in $d \geqslant 3$ and $d = 2$ and is summarised in Tables 17 and 18, respectively. We then investigated whether any central extended graded conformal algebras admit an invariant inner product and we found that this was the case for GCA_{16} (which exists only for $d = 3$) and, provided that $d = 2$, also for GCA_1 and GCA_8.

We then discussed some other notions of conformal algebras obtained by relaxing and/or modifying some of the properties of the graded conformal algebras.

In Section 6 we discussed a class of Lie algebras defined by dropping the condition that D is a grading element in Definition 1. We call the resulting Lie algebras *generalised conformal algebras* (see Definition 2), but it is questionable whether all such Lie algebras are in any way conformal, since all they share with the ur-example of conformal algebra (the algebra of conformal symmetries of Minkowski spacetime) is that they have the generators transform in the same way under the $\mathfrak{so}(d)$ subalgebra of rotations. Nevertheless we present some preliminary results about their classification via deformation theory for $d \geqslant 4$. We will use some of these results below in a restricted context.

In Section 7 we discuss what we call *generalised Lifshitz algebras* (see Definition 3), which consist of a graded kinematical Lie algebra extended by the grading element, but not requiring that the degrees are those of the graded conformal algebras treated in the bulk of this paper. The results here are preliminary, but we present a classification of the possible $\mathbb{Z}$-gradings of all kinematical Lie algebras, which is contained in Table 21.

In Section 8 we discussed a class of conformal Lie algebras related to the Schrödinger algebra. The Schrödinger algebra can be understood as a central extension of a conformal algebra and this suggests the definition of a *generalised Schrödinger algebra* (see Definition 4), whose main property is the existence of an $\mathfrak{so}(d) \oplus \mathfrak{sl}(2,\mathbb{R})$ subalgebra under which the remaining (non-central) generators transform according to $V \otimes E$, where V is the vector representation of $\mathfrak{so}(d)$ and E is a representation of $\mathfrak{sl}(2,\mathbb{R})$ of dimension 2 or 3. We classified the isomorphism classes of generalised Schrödinger algebras with $\dim E = 2$ for $d \geqslant 3$ and with $\dim E = 3$ for $d \geqslant 4$. Apart from the Schrödinger algebra, we find an (non-central) extension by a vector representation of $\mathfrak{so}(d)$ (trivial under $\mathfrak{sl}(2,\mathbb{R})$) and centrally extended algebras where E is the trivial representation of $\mathfrak{sl}(2,\mathbb{R})$.

There are a number of open problems remaining to complete the results in the last three sections, among which we list the following in no particular order:

- Classify the generalised conformal algebras of Section 6. For $d \geqslant 4$ we need only solve the equations (185) and quotient by the action of the automorphisms of the static algebra. For $d \leqslant 3$ the deformation problem has to be looked anew.
- Classify the generalised Schrödinger algebras of Section 8 for $d \leqslant 3$ (for $\dim E = 3$) and for $d = 2$ (for $\dim E = 2$).
- Study the addition of generators to the generalised Lifshitz algebras of Section 7. This is not unrelated to the classification of the generalised conformal algebras where one of the scalar generators is a grading element.
- Classify filtered subdeformations of $\mathfrak{so}(d+1,2)$ or $\mathfrak{so}(d+2,1)$ containing an $\mathfrak{so}(d)$ subalgebra.

A possible extension of this work is to determine the $(d+1)$-dimensional homogeneous manifolds of the corresponding groups, as was done in [9]. This might give a rich class of "spacetimes" in which to discuss conformal-like theories. Of course, of the graded conformal algebras treated in the bulk of the paper, only $\mathsf{GCA}_{15}^{(\varepsilon)}$ can act on a $(d+1)$-dimensional (riemannian or lorentzian) manifold via conformal Killing vectors: their dimension would mean that the manifold is conformally flat and, hence, any such conformal algebra would be isomorphic to $\mathsf{GCA}_{15}^{(\varepsilon)}$ for some value of ε.

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